Takehome Final
Math bootcamp, 2012

Open book. Due 5:00 pm, August 24.
Please drop at 326 Speakman Hall or email to ydong@temple.edu

1. For positive numbers \(a_1, \ldots, a_n\), define

\[
\text{arithmetic mean } a_A = \frac{1}{n} \sum_{i=1}^{n} a_i,
\]

\[
\text{geometric mean } a_G = (a_1 a_2 \ldots a_n)^{1/n},
\]

\[
\text{harmonic mean } a_H = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.
\]

Use Jensen’s inequality to prove that \(a_H \leq a_G \leq a_A\).

2. Prove that for \(a_n = 1/\sqrt{n}\) and \(b_n = 1/n^3\), we have \(\sum_{n=1}^{\infty} a_n\) diverges and \(\sum_{n=1}^{\infty} b_n\) converges.

3. Consider \(f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}\), where \(\mu\) and \(\sigma^2\) are constants.
   a) Check \(\int_{-\infty}^{\infty} f(x)dx = 1\).
   b) Verify that \(E(X) = \int_{-\infty}^{\infty} xf(x)dx = \mu\).
   c) Verify that \(E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \sigma^2 + \mu^2\).
   Hint: use the fact (without proof) \(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx = 1\) and substitute \(x\) with \(y = (x - \mu)/\sigma\).

4. A light is on the top of a 12 ft tall pole and a 5ft 6in tall person is walking away from the pole at a rate of 2 ft/sec.
   a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?
   b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

5. Evaluate the following integration

\[
\int \frac{\sin(1-x)}{2 + \cos^2(1-x)}
\]
6. \(X \sim \text{Binomial}(n, p)\) with
\[
P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \text{ for } x = 0, 1, \ldots, n.
\]
Calculate
\[
E(X^2) = \sum_{x=0}^{n} x^2 P(X = x).
\]

7. When codes messages are sent, there are sometimes errors in transmission. In particular, Morse code uses “dots” and “dashes”, which are known to occur in the proportions of 3 : 4. Suppose there is interference on the transmission line, and with probability 1/8 a dot is mistaken received as a dash, with probability 1/8 a dash is mistaken received as a dot. If we receive a dot, what is the probability of a dot being sent out?

8. Use the alternating series test to determine whether the series
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}
\]
is divergent or convergent (page 11 of the week 3 lecture notes). Then use the Taylor expansion of \(\ln(1 + x)\) to show that
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.
\]

9. Find all the first and second order derivatives of
\[
f(x, y) = \sin(2x) - x^3 e^{5y} + 3y^2
\]

10. Find the limit of sequence \(\{a_n\}\), where \(a_n = n^{1/n}\).
Hint: let \(n^{1/n} = 1 + \delta_n\) and prove that \(\lim_{n \to \infty} \delta_n = 0\).