1 Exponents

For any real number $x$, the powers of $x$ are: $x^0 = 1$, $x^1 = x$, $x^2 = x \cdot x$, etc.

Powers are also called exponents.

Remark: $0^0$ is indeterminate.

Fractional exponents are also called roots. Square root, $x^{1/2} = \sqrt{x}$; cube root, $x^{1/3} = \sqrt[3]{x}$. Generally, $x^{m/n} = \sqrt[n]{x^m}$.

Negative exponent is defined as $x^{-r} = 1/x^r$.

Properties of exponents:

1. $x^a \cdot x^b = x^{a+b}$;
2. $x^a/x^b = x^{a-b}$;
3. $(x^a)^b = x^{ab}$.

2 Functions

We say “$y$ is a function of $x$”, written as $y = f(x)$. This means, for every value of $x$, there is one and only one value of $y$.

$XY$-plane consists of two perpendicular axes, a horizontal $X$-axis and a vertical $Y$-axis. The intersect is the origin $(0,0)$. The plane is partitioned into four disjoint regions called quadrants. Every point $P$ in the plane can be represented by the ordered pair $(x, y)$. The first value is the $x$-coordinate, and indicates its horizontal position relative to the origin. The second value is the $y$-coordinate, and indicates the vertical position relative to the origin.

For a given function $y = f(x)$, the set of all ordered pairs $(x, y)$ that satisfy this equation is called the graph of the function.

Exercise: Draw the graph of $x = -4$, $y = 2$, $y = 2x + 3$, $y = x^2$, $y = x^{1/2}$, $y = 1/x$, $y = 2^x$, $y = |x|$.
The linear function \( y = ax + b \) forms the graph of a line in the XY-plane, where \( b \) is the Y-intercept, and \( a \) is the slope of the line. The slope is the ratio of the height change \( \delta y \) to the length change \( \delta x \), or \( a = \frac{\delta y}{\delta x} = \frac{y_2 - y_1}{x_2 - x_1} \).

Relation between graphs:

1. The graph of \(-f(x)\) is flipping the graph of \(f(x)\) along the \(x\)-axis;
2. The graph of \(f(-x)\) is flipping the graph of \(f(x)\) along the \(y\)-axis;
3. For \(c > 0\), the graph of \(f(x) + c\) is the graph of \(f(x)\) shifted up for \(c\) units;
4. For \(c > 0\), the graph of \(f(x) + c\) is the graph of \(f(x)\) shifted to the left for \(c\) units;
5. For \(c > 0\), the graph of \(f(x) - c\) is the graph of \(f(x)\) shifted to the right for \(c\) units;
6. For \(c > 0\), the graph of \(cf(x)\) and the graph of \(f(x)\) have the same shape and are only different in the \(y\)-scale.
7. For \(c > 0\), the graph of \(f(cx)\) and the graph of \(f(x)\) have the same shape and are only different in the \(x\)-scale.

The quadratic function \( y = ax^2 + bx + c \) (standard form), \( a \neq 0 \) forms the graph of a parabola. It can be “convex” (holds water), such as \( y = x^2 \); or “concave” (spills water), such as \( y = -x^2 \).

By complete of squares, we have

\[
y = ax^2 + bx + c = a \left[ x^2 + \frac{b}{a} x + \left( \frac{b}{2a} \right)^2 \right] + \left( c - \frac{b^2}{4a} \right) = a(x + \frac{b}{2a})^2 + \left( c - \frac{b^2}{4a} \right) \]

1. From the above vertex form, we see the axis of symmetry is \( x = -\frac{b}{2a} \);
2. If \( a > 0 \), it’s convex; if \( a < 0 \), it’s concave;
3. The discriminant is defined to be $\Delta = b^2 - 4ac$. If $\Delta < 0$, then the polynomial $ax^2 + bx + c$ has no real roots; if $\Delta = 0$, then the polynomial has one real root; if $\Delta > 0$, then the polynomial has two real roots

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$  

It is useful that $x_1 + x_2 = -b/a$ and $x_1x_2 = c/a$.

The function $y = 1/x$ forms the graph of hyperbola, which has two symmetric branches.

Exercise: Consider general power functions $y = x^p$ with different constant $p$.

The function $y = 2^x$ is not a power function. Its graph is an exponential growth curve, which doubles in height with each unit increase in $x$. The graph of $y = 2^{-x}$ is an exponential decay curve.

The most useful exponential function is $y = \exp(x) = e^x$, where the base $e = 2.71828\ldots$ is the Euler’s constant.

3 Injection, surjection and bijection

**Domain** of a function is the set of “input” or argument values for which the function is defined.

**Codomain** of a function is the set into which all of the output of the function is constrained to fall.

Let $f$ be a function whose domain is a set $X$. It is an **injection**, then for all $a$ and $b$ in $X$, if $f(a) = f(b)$, then $a = b$.

Let $f$ be a function with domain $X$ and codomain $Y$. It is an **surjection**, then for every $y$ in the codomain $Y$ there is at least one $x$ in the domain $X$ such that $f(x) = y$.

If $f$ is both an injection and a surjection, then it is a **bijection**, or one-to-one correspondence.

If there is a one-to-one correspondence between $x$ and $y$, $x = f^{-1}(y)$ is the inverse function of $y = f(x)$.

$$x \xrightarrow{f} y \xrightarrow{f^{-1}} x \text{ and } y \xrightarrow{f^{-1}} x \xrightarrow{f} y.$$
If we draw the graph of a function and its inverse function together, they are going to be symmetric about the diagonal line \( y = x \).

The composition of two functions is defined to be \( f \cdot g(x) = f(g(x)) \). Thus we have \( f \cdot f^{-1}(x) = f^{-1} \cdot f(x) = x \).

The inverse function of \( y = b^x \) is the logarithm function \( y = \log_b(x) \). For example, we have

\[
10^2 = 100, \quad 9^{1/2} = 3, \quad 99^0 = 1, \quad 5^{-1} = 0.2; \\
\log_{10}(100) = 2, \log_9 3 = 1/2, \log_{99}(1) = 0, \log_5 0.2 = -1.
\]

The choice of base \( b = 10 \) is called the “common logarithm” \( \log_{10}(x) \); the base \( b = e \) is the “natural logarithm” \( \log_e(x) \), and is usually denoted by \( \ln(x) \). Then we have \( e^{\ln(x)} = x \) and \( \ln(e^x) = x \).

Basic properties of logs

1. \( \log_b(xy) = \log_b x + \log_b y \);
2. \( \log_b(x/y) = \log_b x - \log_b y \);
3. \( \log_b(x^n) = n \log_b x \);
4. \( \log_b x = \log_a x / \log_a b \).

As a result of 4, we have \( \log_a a = 1 / \log_a b \).

4 Limits and derivatives

We have seen that as \( x \) becomes larger, the value of \( 1/x \) become smaller. It can not reach 0, but we can make \( 1/x \) as close to 0 as possible by making \( x \) large enough. This is a limit statement:

\[
\lim_{x \to \infty} \frac{1}{x} = 0.
\]

The \((\epsilon, \delta)\)-definition of the limit of a function is as follows: Let \( f \) be a function defined on an open interval containing \( c \) and let \( L \) be a real number. Then

\[
\lim_{x \to c} f(x) = L
\]
means for each real \( \epsilon > 0 \), there exists a real \( \delta > 0 \) such that for all \( x \) with \( 0 < |x - c| < \delta \), we have \(|f(x) - L| < \epsilon\). Symbolically,

\[ \forall \epsilon > 0 \exists \delta > 0 : \forall x (0 < |x - c| < \delta \implies |f(x) - L| < \epsilon). \]

The first order derivative of a function \( y = f(x) \) is

\[ \frac{dy}{dx} = f'(x) = \lim_{\delta \to 0} \frac{\delta y}{\delta x} = \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta}. \]

The process of calculate the derivative is called differentiation. We call \( dy \) and \( dx \) differentials. Geometrically, \( f'(x_0) \) is the instant rate of change at point \((x_0, f(x_0))\), or the slope of the tangent line to the curve \( y = f(x) \) at point \((x_0, f(x_0))\).

For example, \( y = x^2 \), then

\[ \frac{dy}{dx} = \lim_{\delta \to 0} \frac{(x + \delta)^2 - x^2}{\delta} = \lim_{\delta \to 0} (2x + \delta) = 2x. \]

Generally, we have the power rule:

\[ \frac{dx^p}{dx} = px^{p-1}. \]

We have \( \frac{dx}{dx} = e^x \), or the derivative of \( e^x \) is itself. The general exponential rule is:

\[ \frac{db^x}{dx} = b^x \ln(b). \]

We have \( \frac{d[\ln(x)]}{dx} = \frac{1}{x} \). The general logarithm rule is:

\[ \frac{d[\log_b(x)]}{dx} = \frac{1}{x \ln(b)} \]

If \( y = f(x) = c \) is a constant for any \( x \), then \( f'(x) = 0 \) for all \( x \).

Not every function has a derivative everywhere!

Properties of derivatives:

1. For any constant \( c \) and differentiable function \( f(x) \), \([cf(x)]' = cf'(x)\);

For any two differentiable functions \( f(x) \) and \( g(x) \)

2. Sum and difference rule, \([f(x) \pm g(x)]' = f'(x) \pm g'(x)\);
3. Product rule, \( [f(x)g(x)]' = f'(x)g(x) + f(x)g'(x); \)

4. Quotient rule, \( \left[ \frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \) for \( g(x) \neq 0; \)

5. Chain rule, \( \{f[g(x)]\}' = f'[g(x)]g'(x); \)

6. If \( f(x) \) is the inverse function of \( g(x) \), then \( g'(x) = \frac{1}{f'(g(x))}. \)

Problems from the pre-exam
1. Find the derivatives of the following functions:
   a) \( f(x) = (6x^3 - x)(10 - 20x). \)
   b) \( f(x) = \frac{3x+9}{2-x}. \)
   c) \( f(x) = a^x. \)

2. Determine whether the following statements are right or wrong:
   a) \( \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0 \) and \( \lim_{\theta \to 0} \frac{\theta}{\sin(\theta)} = 1. \)
   b) \( \lim \cos(\pi/\theta) \) does not exist.
   c) \( \lim_{x \to 0^+} \frac{1}{x} = \infty \) and \( \lim_{x \to 0^+} \frac{1}{x^2} \) does not exist.

3. Find equation of tangent line to \( f(x) = 4x - 8\sqrt{x} \) at \( x = 16. \)

4. Determine \( f'(0) \) for \( f(x) = |x|. \)

Examples from class
Find \( y' \) for a) \( y = x^x; \) b) \( x^3y^5 + 3x = 8y^3 + 1. \)

Please also read more.pdf, pages 7 to 11. The additional material is due to Prof. Ismor Fischer from University of Wisconsin Madison (http://pages.stat.wisc.edu/ ifischer/).
5 Applications

5.1 Newton-Raphson method

Suppose we want to solve for the roots of equation \( f(x) = 0 \).

1. We start with initial guess \( x_0 \).
2. Calculate \( x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \), for \( i = 0, 1, \ldots \).
3. Stop the iteration until \( |x_{i+1} - x_i| < \epsilon \), where \( \epsilon \) is a prespecified small positive number, say \( 10^{-4} \). Otherwise go back to Step 2.

Exercise: Solve for \( x \): \( f(x) = x^3 - 21x^2 + 135x - 220 = 0 \). Hint: Check that \( f(2) = -26 \) and \( f(3) = 23 \). Use initial value \( x_0 = 2 \).

Note: Solve for \( x^{1/3} = 0 \) using Newton-Raphson. We will get \( x_{i+1} = -2x_i \), and the algorithm will not converge. This means Newton-Raphson may not always work!

5.2 Local extrema

If a function \( f(x) \) has either a local maximum or a local minimum at some value \( x_0 \), and \( f(x) \) is differentiable at \( x_0 \), then the tangent line must be horizontal, or \( f'(x_0) = 0 \). We call \( x_0 \) a local extrema if it is either a local maximum or a local minimum. On the other hand, not all solutions of \( f'(x) = 0 \) is a local extrema.

Properties:

1. If \( f(x) \) has a local extrema at \( x = x_0 \) and \( f(x) \) is differentiable at \( x_0 \), then \( f''(x_0) = 0 \).
2. If \( f'(x) > 0 \) for every \( x \) on some interval \( I \), then \( f(x) \) is increasing on the interval.
3. If \( f'(x) < 0 \) for every \( x \) on some interval \( I \), then \( f(x) \) is decreasing on the interval.
4. If \( f'(x) = 0 \) for every \( x \) on some interval \( I \), then \( f(x) \) is constant on the interval.
5. If \( f''(x) > 0 \) for all \( x \) on some interval \( I \), then \( f(x) \) is concave up on \( I \).
6. If $f''(x) < 0$ for all $x$ on some interval $I$, then $f(x)$ is concave down on $I$.

7. If $f'(x_0) = 0$, $f''(x_0) < 0$ and $f''(x)$ is continuous in a region around $x = x_0$, then $x = x_0$ is a local maximum.

8. If $f'(x_0) = 0$, $f''(x_0) > 0$ and $f''(x)$ is continuous in a region around $x = x_0$, then $x = x_0$ is a local minimum.

9. If $f'(x_0) = 0$, $f''(x_0) = 0$ and $f''(x)$ is continuous in a region around $x = x_0$, then $x = x_0$ can be a local maximum, a local minimum, or neither.

Exercise: Study $f(x) = x^3 - 21x^2 + 135x - 220 = 0$. Calculate the first and second order derivative at $x = 5, 6, 7, 8, 9$. Describe what you find.

### 5.3 Taylor series

Derivative of the derivative is called the second-order derivative

$$f^{(2)}(x) = f''(x) = \left[ f'(x) \right]' = \frac{d}{dx} \left( \frac{dy}{dx} \right).$$

Similarly, we can define the $n$-th order derivative and denote it by $f^{(n)}$.

The Taylor series of a function $f(x)$, which is infinitely differentiable in a neighborhood of $a$, is a power series:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Some important series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \text{ for } -1 < x \leq 1$$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1$$
5.4 Some examples

Example 1: Air is being pumped into a spherical balloon at a rate of 5 cm$^3$/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Solution: Both the volume of the balloon and the radius of the balloon will vary with time and so are functions of time. Denote them as $V(t)$ and $r(t)$ respectively. Then we have

$$V(t) = \frac{4}{3}\pi r^3(t).$$

Take derivatives with respect to $t$ on both sides and we get

$$V'(t) = \frac{4}{3}\pi 3r^2(t)r'(t).$$

Plug in $V'(t) = 5$ and $r(t) = 20/2 = 10$, we get

$$r'(t) = \frac{V'(t)}{4\pi} r^2(t) = \frac{5}{4\pi \times 10^2} = \frac{1}{80\pi} \text{cm/min.}$$

Example 2: A 15 foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of 1/4 ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing?

Solution: Define the distance of the bottom of the ladder from the wall to be $x(t)$ and the distance of the top of the ladder from the floor to be $y(t)$. Because $x(t)$ decreases as $t$ increases, we have $x'(t) = -1/4$. We also have

$$x^2(t) + y^2(t) = 15^2.$$

Take derivatives with respect to $t$ on both sides and we get

$$2x(t)x'(t) + 2y(t)y'(t) = 0,$$

and it follows that $y'(t) = -x(t)x'(t)/y(t)$. We have at $t = 12$,

$$x(t) = 10 - 12 \times 1/4 = 7, \text{ and } y(t) = \sqrt{15^2 - x^2(t)} = \sqrt{15^2 - 7^2} = \sqrt{16}. $$

Together we have

$$y'(t) = -x(t)x'(t)/y(t) = -7 \times (-1/4)/\sqrt{16} = 0.1319 \text{ft/sec.}$$
Example 3: A spotlight is on the ground 20 ft away from a wall and a 6 ft tall person is walking towards the wall at a rate of 2.5 ft/sec. How fast is the height of the shadow changing when the person is 8 feet from the wall? Is the shadow increasing or decreasing in height at this time?

Solution: Define the distance of the person to the spotlight to be \( x(t) \) and the height of the shadow to be \( y(t) \). Then we have

\[
\frac{x(t)}{20} = \frac{6}{y(t)}, \quad \text{or} \quad y(t) = \frac{120}{x(t)}.
\]

Take derivatives with respect to \( t \) on both sides and we get

\[
y'(t) = -\frac{120x'(t)}{x^2(t)}.
\]

Because \( x(t) \) increases as \( t \) increases, we have \( x'(t) = 2.5 \). We also have \( x(t) = 20 - 8 = 12 \). Together we have

\[
y'(t) = -120 \times 2.5/12^2 = -2.083 \text{ ft/sec}.
\]

The negative sign means that \( y(t) \) decreases as \( t \) increases, or the shadow becomes shorter as the person walks closer to the wall.

Example 4: An apartment complex has 250 apartments to rent. If they rent \( x \) apartments then their monthly profit, in dollars, is given by,

\[
P(x) = -8x^2 + 3200x - 80000.
\]

How many apartments should they rent in order to maximize their profit?

Solution: \( P'(x) = -16x + 3200 = -16(x - 200) \). Thus if \( x < 200 \), \( P'(x) > 0 \) and \( P(x) \) increases. If \( x > 200 \), \( P'(x) < 0 \) and \( P(x) \) decreases. This means for \( 0 \leq x \leq 250 \), \( P(x) \) becomes the largest when \( x = 200 \). Thus 200 apartments should be rented to maximize the profit.

Example 5: Determine all the intervals where the following function is increasing or decreasing

\[
f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5.
\]

Solution: Take first order derivative and we get

\[
f'(x) = -5x^4 + 10x^3 + 40x^2 = -5x^2(x - 4)(x + 2).
\]
Then it is easy to see that $f'(x) > 0$, thus $f(x)$ increases if $-2 < x < 0$ or $0 < x < 4$. $f'(x) < 0$, thus $f(x)$ decreases if $x < -2$ or $x > 4$.

**Example 6:** A light is on the top of a 12 ft tall pole and a 5ft 6in tall person is walking away from the pole at a rate of 2 ft/sec.

a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

**Solution:** a. Define the distance of the person from the pole to be $x_p$, the length of the shadow as $x_s$. Then we have

$$\frac{x_s}{x_s + x_p} = \frac{5.5}{12} \quad \text{and} \quad x_s = \frac{11x_p}{13}.$$  

Define the distance between the tip of the shadow and the pole as $x$. Then $x = x_p + x_s = x_p + \frac{11x_p}{13} = \frac{24x_p}{13}$. It follows that

$$x' = \frac{24}{13} x'_p = \frac{24}{13} \times 2 = \frac{48}{13} \text{ ft/sec}.$$  

b. Because $x = x_p + x_s$, we have

$$x' = x'_p + x'_s \quad \text{and} \quad x'_s = x' - x'_p = \frac{48}{13} - 2 = \frac{22}{13}.$$  

## 6 Integrals

Suppose we have a general function $y = f(x)$. For simplicity, let $f(x) > 0$ and $f(x)$ continuous. Denote

$$F(x) = \text{area under the graph of } f \text{ in the interval } [a,x].$$  

Then we have, for some value $z$ in the interval $[x, x + \delta]$

$$F(x + \delta) - F(x) = f(z) \delta,$$

or

$$\frac{F(x + \delta) - F(x)}{\delta} = f(z).$$  

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As $\delta$ goes to 0, $z$ goes to $x$, and we have

$$F'(x) = \lim_{\delta \to 0} \frac{F(x + \delta) - F(x)}{\delta} = \lim_{z \to x} f(z) = f(x).$$

So $F$ is antiderivative of $f$, and we denote

$$F(x) = \int_a^x f(t) dt.$$

This is a definite integral of the function $f$ from $a$ to $x$. $f$ is called the integrand. We also have indefinite integral. That is, for an arbitrary constant $C$,

$$\int f(x) = F(x) + C.$$

Properties of integrals:

1. For any constant $c$ and any integrable function $f(x)$,

$$\int c f(x) dx = c \int f(x) dx.$$

2. For any two integrable functions $f(x)$ and $g(x)$,

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx.$$

3. **Integration by parts.** Given the existence of all integrations, we have

$$\int_a^b f(x) g'(x) dx = f(x) g(x) \bigg|_a^b - \int_a^b f'(x) g(x) dx.$$

**Example 1:** \( \int \frac{3y}{5y^2 + 4} dy \)

**Solution:** \( \int \frac{3y}{5y^2 + 4} dy = \frac{3}{10} \int \frac{10y}{5y^2 + 4} dy = \frac{3}{10} \int \frac{1}{5y^2 + 4} d(5y^2 + 4) = \frac{3}{10} \ln(5y^2 + 4) + c. \)

**Example 2:** \( \int \frac{3}{5y^2 + 4} dy \)

**Sol:** \( \int \frac{3}{5y^2 + 4} dy = \int \frac{3/4}{5y^2/4 + 1} dy = \frac{3}{4} \int \frac{1}{(\sqrt{5}/2y)^2 + 1} d(\sqrt{5}/2y) = \frac{3}{2\sqrt{5}} \arctan(\sqrt{5}/2y) + c \)

**Example 3:** \( \int_1^0 e^{-x} dx, \int_0^x xe^{-x} dx, \int_0^1 x^2 e^{-x} dx \)
\begin{align*}
\text{Sol: } & \int_1^0 e^{-x}dx = -\int_0^1 e^{-x}d(-x) = -e^{-x}|_0^1 = 1 - e^{-1} \\
& \int_1^0 xe^{-x}dx = -\int_0^1 xde^{-x} = -xe^{-x}|_0^1 - (-\int_0^1 e^{-x}dx) = -e^{-1} + 1 - e^{-1} = 1 - 2e^{-1} \\
& \int_1^0 x^2 e^{-x}dx = -\int_0^1 x^2de^{-x} = -x^2e^{-x}|_0^1 - (-\int_0^1 e^{-x}2xdx) = -e^{-1} + 2(1 - 2e^{-1}) = 2 - 5e^{-1} \\
\end{align*}

Example 4: Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.
\textbf{Sol:} Draw the graphs of two functions in the same figure. Then we see the area equals
\[ \int_1^0 (\sqrt{x} - x^2)dx = \left( \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right)|_1^0 = \frac{1}{3}. \]

Example 5: Determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt[3]{x}$, $x = 8$, and the x-axis about the x-axis.
\textbf{Sol:} Draw a horizontal line, which will be rotated to a cylinder. We have the volume equals
\[ \int_0^8 2\pi y(8 - y^3)dy = \frac{96}{5}\pi. \]
Or we can draw a vertical line, which will be rotated to a circle. We have the volume equals
\[ \int_0^8 \pi(\sqrt[3]{x})^2dx = \frac{96}{5}\pi. \]

7 Partial fractions

Example 1: Determine $\int_4^8 \frac{3x+11}{x^2-x-6}dx$.
\textbf{Sol:} Because $x^2 - x - 6 = (x - 3)(x + 2)$, let
\[ \frac{3x+11}{x^2-x-6} = \frac{a}{x-3} + \frac{b}{x+2} = \frac{a(x+2) + b(x-3)}{(x-3)(x+2)} \]
Solve for $a$ and $b$ from
\[ \begin{cases} 
\frac{a+b}{x-3} = 3 \\
\frac{2a-3b}{x+2} = 11 
\end{cases} \]
and we get $a = 4$, $b = -1$. Thus
\[ \int_4^8 \frac{3x+11}{x^2-x-6}dx = \int_4^8 \left( \frac{4}{x-3} - \frac{1}{x+2} \right)dx \\
= \left( 4\ln(x-3) - \ln(x+2) \right)|_4^8 = 4\ln5 - \ln10 + \ln6 \]

Example 2: Determine $\int_2^3 \frac{x^2}{x^2-1}dx$. 

Sol:

\[
\frac{1}{x^2 - 1} = \frac{a}{x - 1} + \frac{b}{x + 1} = \frac{a(x + 1) + b(x - 1)}{x^2 - 1}
\]

Solve for \(a\) and \(b\) from

\[
\begin{cases}
  a + b = 0 \\
  a - b = 1
\end{cases}
\]

and we get \(a = 1/2, b = -1/2\). Thus

\[
\int_2^3 \frac{x^2}{x^2 - 1} dx = \int_2^3 \left( \frac{x^2 - 1}{x^2 - 1} + \frac{1}{x^2 - 1} \right) dx = \int_2^3 \left( \frac{x^2 - 1}{x^2 - 1} + \frac{1/2}{x - 1} + \frac{-1/2}{x + 1} \right) dx
\]

\[
= \left( x + \frac{1}{2} ln(x - 1) - \frac{1}{2} ln(x + 1) \right) \bigg|_2^3 = 1 + \frac{1}{2} ln2 - \frac{1}{2} ln4 + \frac{1}{2} ln3
\]

8 Improper integrals

We have the following properties

1. \(\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx\) if the limit on the right hand side exists and is finite, in which case we say the integral converges. Otherwise we say the integral diverges.

2. \(\int_{-\infty}^a f(x)dx = \lim_{t \to -\infty} \int_t^a f(x)dx\) if the limit on the right hand side exists and is finite, in which case we say the integral converges. Otherwise we say the integral diverges.

3. \(\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx\) if both integrals on the right hand side converge.

Example 1: For \(a > 0\), consider

\[
\int_a^\infty \frac{1}{x^p} dx.
\]

Then the integral converges if \(p > 1\), and diverges if \(p \leq 1\).

Example 2: Does the following integral diverge or converge?

\[
\int_{-\infty}^0 \frac{1}{\sqrt{3 - x}} dx.
\]
\[ \int_{-\infty}^{0} \frac{1}{\sqrt{3-x}} \, dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{\sqrt{3-x}} \, dx = -2\sqrt{3} + \lim_{t \to -\infty} 2\sqrt{3-t}. \]

Thus the integral diverges as the limit goes to positive infinity.

**Example 3:** Does the following integral diverge or converge?

\[ \int_{2}^{\infty} \sin x \, dx. \]

**Sol:**

\[ \int_{2}^{\infty} \sin x \, dx = \lim_{t \to \infty} \int_{2}^{t} \sin x \, dx = \lim_{t \to \infty} (-\cos t + \cos 2). \]

Thus the integral diverges as the limit does not exist.

**Comparison test:** For \( f(x) \geq g(x) \geq 0 \)

1. if \( \int_{a}^{\infty} f(x) \, dx \) converges, then \( \int_{a}^{\infty} g(x) \, dx \) converges.
2. if \( \int_{a}^{\infty} g(x) \, dx \) diverges, then \( \int_{a}^{\infty} f(x) \, dx \) diverges.

**Example 4:** Does the following integral diverge or converge?

\[ \int_{2}^{\infty} \frac{\cos^2 x}{x^2} \, dx. \]

**Sol:** Let \( f(x) = 1/x^2 \) and \( g(x) = \cos^2 x/x^2 \). Then \( f(x) \geq g(x) \geq 0 \) and \( \int_{2}^{\infty} f(x) \, dx = \int_{2}^{\infty} 1/x^2 \, dx \) converges. It follows from the comparison test that \( \int_{2}^{\infty} \cos^2 x/x^2 \, dx = \int_{2}^{\infty} g(x) \, dx \) converges as well.

**Example 5:** Does the following integral diverge or converge?

\[ \int_{3}^{\infty} \frac{1}{x + e^x} \, dx. \]

**Sol:** Let \( g(x) = 1/(x + e^x) \) and \( f(x) = 1/e^x \). Then \( f(x) \geq g(x) \geq 0 \) for \( x > 3 \). Consider

\[ \int_{3}^{\infty} \frac{1}{e^x} \, dx = -e^{-x}|_{3}^{\infty} = e^{-3} \]

It follows from the comparison test that \( \int_{3}^{\infty} 1/(x + e^x) \, dx \) converges.
9 Partial derivatives

Example: Find all the first and second order derivatives for
\[ f(x, y) = \cos(2x) - x^2e^{5y} + 3y^2 \]

Sol: The first order partial derivatives are
\[ \frac{\partial}{\partial x} f(x, y) = -2\sin(2x) - 2xe^{5y} \]
\[ \frac{\partial}{\partial y} f(x, y) = -5x^2e^{5y} + 6y \]

The second order partial derivatives are
\[ \frac{\partial^2}{\partial x^2} f(x, y) = \frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x} f(x, y) \right\} = -4\cos(2x) - 2e^{5y} \]
\[ \frac{\partial^2}{\partial x\partial y} f(x, y) = \frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial y} f(x, y) \right\} = -10xe^{5y} \]
\[ \frac{\partial^2}{\partial y\partial x} f(x, y) = \frac{\partial}{\partial y} \left\{ \frac{\partial}{\partial x} f(x, y) \right\} = -10xe^{5y} \]
\[ \frac{\partial^2}{\partial y^2} f(x, y) = \frac{\partial}{\partial y} \left\{ \frac{\partial}{\partial y} f(x, y) \right\} = -25x^2e^{5y} + 6 \]

10 Double integrals

Example: Find the double integration over \( A = [2, 4] \times [1, 2] \)
\[ \int_A \int 6xy^2 dxdy \]

Sol: If we first integrate with respect to \( x \), then we have
\[ \int_2^4 \left( \int_2^4 6xy^2 \right) dy = \int_2^4 \left( 6y^2x^2 \right) \bigg|_{x=2}^{x=4} dy = 36y^2 \bigg|_{y=1}^{y=2} = 72 = 84 \]

Or we can first integrate with respect to \( y \) and get
\[ \int_2^4 \left( \int_2^4 6xy^2 \right) dx = \int_2^4 \left( 6x^2y^2 \right) \bigg|_{x=1}^{x=4} dx = 14x^2 \bigg|_{x=1}^{x=2} = 84 \]