1. Find the Taylor expansion of \( f(x) = e^{-x} \) at \( x = 0 \).

2. Let \( P(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \), for \( x = 0, 1, \ldots, n \). Calculate

\[
E(X^2) = \sum_{x=0}^{n} x^2 P(X = x).
\]

3. For positive numbers \( a_1, \ldots, a_n \), define

- arithmetic mean \( a_A = \frac{1}{n} \sum_{i=1}^{n} a_i \),
- geometric mean \( a_G = (a_1 a_2 \ldots a_n)^{1/n} \),
- harmonic mean \( a_H = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \).

Use Jensen’s inequality to prove that \( a_H \leq a_G \leq a_A \).

4. Prove that for \( x_i > 0, i = 1, \ldots, n \) satisfying \( \prod_{i=1}^{n} x_i = x_1 x_2 \cdots x_n = 1 \), we have

\[
\prod_{i=1}^{n} (1 + x_i) = (1 + x_1)(1 + x_2) \cdots (1 + x_n) \geq 2^n.
\]

Hint: first prove that \( 1 + x_i \geq 2\sqrt{x_i} \).

5. Prove that for any \( x > 0, y > 0 \) and \( 0 < \alpha < 1 \), we have

\[
(x + y)\alpha < x^\alpha + y^\alpha.
\]

Hint: let \( z = x/y \). Consider \( f(z) = (z + 1)^\alpha - z^\alpha - 1 \) at \( z = 0 \) and \( f'(z) \) at \( z > 0 \).
6. Evaluate the following integrations

\[\int \frac{x}{1 + x^2} dx, \int \frac{1}{1 - x^2} dx, \int_0^1 \frac{1}{1 + x^2} dx, \int_0^{1/2} \frac{1}{\sqrt{1 - x^2}} dx.\]

Hint: use the fact that \(\int \frac{1}{x} dx = \ln|x| + c\).

7. For \(g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\), suppose we know \(\int_{-\infty}^{\infty} g(x) dx = 1\), \(\int_{-\infty}^{\infty} g(x) x dx = 0\) and \(\int_{-\infty}^{\infty} g(x) x^2 dx = 1\). Consider \(f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/(2\sigma^2)}\), where \(\mu\) and \(\sigma^2\) are known constants.

a) Verify that \(\int_{-\infty}^{\infty} f(x) dx = 1\).

b) Verify that \(E(X) = \int_{-\infty}^{\infty} x f(x) dx = \mu\).

c) Verify that \(E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \sigma^2 + \mu^2\).

Hint: change of variable with \(y = (x-\mu)/\sigma\).

8. For nonzero numbers \(a\), \(b\), \(c\) and \(d\) satisfying \(ad \neq bc\), consider matrix

\[A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\]

a) Find \(\det(A)\) and \(A^{-1}\);

b) Prove that \(AA^T\) is positive definite.

Hint: suppose \(x_1\) and \(x_2\) are solutions of quadratic equation \(ax^2 + bx + c = 0\) with \(a \neq 0\), then \(x_1 + x_2 = -b/a\) and \(x_1x_2 = c/a\).

9. Find the maximum and minimum of \(f(x, y) = 5x - 3y\) subject to the constraint that \(x^2 + y^2 = 136\).

10. Urn 1 contains 5 white balls and 7 black balls. Urn 2 contains 3 whites and 12 black. A fair coin is flipped; if it is Heads, a ball is drawn from Urn 1, and if it is Tails, a ball is drawn from Urn 2. Suppose that this experiment is done and you learn that a white ball was selected. What is the probability that this ball was in fact taken from Urn 2? (i.e., that the coin flip was Tails)