ANALYSIS OF STOCHASTIC GROUNDWATER PROBLEMS. PART I: DETERMINISTIC PARTIAL DIFFERENTIAL EQUATIONS IN GROUNDWATER FLOW. A FUNCTIONAL-ANALYTIC APPROACH

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ABSTRACT


Functional-analytic theory is introduced as a method to analyse the problem of deterministic partial differential equations of the type appearing in groundwater flow. Equations are treated as abstract evolution equations for elliptic partial differential operators in an appropriate functional Sobolev space, for which a strongly continuous semigroup exists. A semigroup solution is then developed and its application to regional groundwater flow is indicated. The theoretical framework presented in Part I serves as introduction for the stochastic groundwater flow problem presented in Part II and gives an unified conception of the deterministic and the stochastic problem.

1. INTRODUCTION

Theory and solutions of ordinary stochastic differential equations in engineering systems are becoming increasingly useful and several works have been published on the topic (see for example Soong, 1973). However, similar descriptions and applications for stochastic partial differential equations (stochastic PDE) are still very rare in the hydrologic literature (see Sagar, 1979). The main difficulties in the presentation of such a theory lie in the limitations of classical mathematics in treating equations with unbounded operators, such as the ones found in PDE, subject to random disturbances, for which a mathematical representation is difficult to obtain in the classical sense.

The objective of the present paper (Part I) is to present the fundamental concepts of functional analysis related to deterministic partial differential equations (deterministic PDE), which will later help us develop a theoretical framework to analyse and solve stochastic PDE in the second paper (Part II, this volume). The meaning and use of these concepts will be illustrated in a simple application to regional groundwater flow, which will be used for
comparison with the equivalent stochastic problem in Part II. Overall, the aim of Part I and Part II is to show how the concepts of functional analysis, abstract evolution equations and semigroups of partial differential operators in appropriate Sobolev spaces can be blended and used to present an integrated and rigorous theory of stochastic PDE and as a practical tool to solve some applications to stochastic groundwater flow problems. The main applications cover the case of a stochastic PDE subject to a random forcing term and that of a stochastic PDE subject to random initial condition. In the third paper (Part III, this volume) we will see how the combination of functional analysis and a formulation of the Ito's lemma in Hilbert spaces may be combined to develop deterministic PDE satisfying the moments of the original stochastic PDE. The main applications of this method cover the analysis and solution of stochastic PDE occurring in two-dimensional or three-dimensional domains of any geometrical shape.

2. SOBOLEV SPACES

A fundamental difficulty in the determination of a solution of a differential equation (selection of an appropriate element of a function space) is the fact that differential operators are unbounded and are not defined on the whole of the Banach spaces $C[a, b]$ or $L_2[a, b]$ (see also the list of symbols). An unbounded operator is one whose norm approaches infinity. Basically, a Banach space is an $n$-dimensional vector space with a norm. A norm refers to the concept of "size" of the element of a space. The norm of a three-dimensional vector space is the absolute value of a vector, which is equal to the square root of the sum of the square of its components. Norms may be defined in many different ways (see Kolmogorov and Fomin, 1970; Griffel, 1981). $C[a, b]$ is the space of continuous functions in the closed interval determined by the points $a$ and $b$ in a one-dimensional space. $L_2[a, b]$ is the space of square-integrable functions in the closed interval $[a, b]$.

Hence the problem of solutions of PDE may be solved by changing the norm, but the resulting space will then be incomplete. To avoid this problem it is possible to use Green's functions to replace the differential equations by integral equations, which involves continuous and compact operators in Banach spaces (see Griffel, 1981 and Oden, 1977 for the abstract concept of compactness). However a distributional approach to $L_2$ briefly described in Serrano (1985) leads to a theory in which differential equations fit naturally and can be treated directly without reduction to integral equations. The theory of distributions is essentially a new foundation of mathematical analysis. It replaces functions of a real variable by new objects that have the same characteristics or ordinary functions, but allow a mathematical representation of singular functions like the dirac delta function. Basically, distributions are defined by a theorem which states that to every locally
integrable function \( f(x) \) there corresponds a distribution \( f \) defined by the inner product (Griffel, 1981):

\[
(f, \phi) = \int_{-\infty}^{\infty} f(x)\phi(x) \, dx
\]

where \( \phi \in C_0^\infty \) is a "smooth" function, that is \( \phi \) is an element of the space of continuous, infinite differentiable (for the superscript \( \infty \)) functions which vanish at the boundary of the region in consideration (for the subscript 0).

While spaces of differentiable ordinary functions are incomplete, spaces of differentiable generalized functions are complete and the theories of Banach and Hilbert spaces are available. A complete space is a more useful one, because it contains all the limits of sequences of approximations to all of its functional elements.

These spaces of differentiable generalized functions are called Sobolev spaces. Generally Sobolev spaces are special cases of Hilbert spaces. A Hilbert space is a Banach space of infinite dimensions with an inner product specified by its norm, which reminds us of the scalar product in vector spaces.

Let us consider an \( n \)-dimensional region \( G \), which may be \( \mathbb{R}^n \) itself or may be the region inside some surface. Thus \( H^1(G) \) is a space of distributions on \( G \) with the norm \( \| \cdot \|_1 \):

\[
\| f \|_1 = \left( \int_G \left( |f|^2 + \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right|^2 \right) \, d^n x \right)^{1/2} \tag{1}
\]

The Sobolev space \( H^1(G) \) is the set of all \( f \in L_2(G) \) such that all the first partial derivatives \( \partial f / \partial x_i \) belong to \( L_2(G) \). Note that a widely used notation for Sobolev spaces defines \( W_p^m \) as the \( m \)-th order Sobolev space, where \( p \) is the power of the space norm and \( m \) is the order of the derivative inside the norm. Hence \( W_2^m = H^m \) in the notation used here, and \( W_2^0 = H^0 = L_2 \). The inner product of \( H^1(G) \) is:

\[
(f, g)_1 = \int_G \left[ (fg) + (\nabla f \cdot \nabla g) \right] \, d^n x \tag{2}
\]

where \( \nabla f \cdot \nabla g \) denotes \( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \). This inner product gives the norm (1). If we denote the \( L_2 \) inner product by a subscript zero:

\[
(f, g)_0 = \int_G fg \, d^n x \tag{3}
\]

then eqn. (2) becomes:

\[
(f, g)_1 = (f, g)_0 + (\nabla f, \nabla g)_0 \tag{4}
\]

\( C_0^\infty(G) \) is dense in \( L_2(G) \) with the zero norm, defined as:

\[
\| f \| = \int_G |f|^2 \, d^n x \tag{5}
\]
because $L_2(G)$ is the Hilbert space obtained by completing the set of smooth functions $C^\infty_0(G)$ with respect to the $L_2$ norm. The abstract concept of a dense space in another refers to a subspace or subset whose boundaries are completely inside and nearby the corresponding boundaries of the involving set (see Griffel, 1981 for a rigorous definition). In the same way $C^\infty(G)$ is dense in $H^1(G)$ with the 1-norm (the norm specified in eqn. 2) and $H^1(G)$ is a Sobolev space, which is itself a Hilbert space.

Now $H^1_0(G)$ is the closure of $C^\infty_0(G)$ in the Hilbert space $H^1(G)$. This subspace is very useful in the theory of PDE because it consists of distributions which vanish at the boundary of $G$, and thus satisfies the boundary conditions of the Dirichlet type often associated with groundwater flow.

We shall identify a distinction between a classical solution and a weak solution of a PDE.

Let $G$ be bounded and have a $C^\infty$ boundary, and let $A$ be a formal partial differential operator given by:

$$Af = \sum_{|k|, |l| \leq m} (-1)^{|k|} D^k (p_{kl} D^l f)$$

where $A$ is assumed elliptic; $p_{kl}$ are real-valued coefficients and $D$ represents differentiation in Hilbert space. For a given $g$ in $C(G)$, a function $f$ is said to be a classical solution of $Af = g$ iff $f \in C^2_0(G)$ and:

$$Af = g \quad \text{in } G$$

For given $g \in L_2$ a function $f \in L_2$ is said to be a weak solution of $Af = g$ if:

$$(f, A^* \phi)_0 = (g, \phi)_0$$

for all $\phi \in C^\infty_0$; $A^*$ is the adjoint operator of $A$. Then we write:

$$Af = w g$$

where the subindex w means "weakly".

A typical example of the above is the Poisson equation:

$$\nabla^2 f = g \quad f, g \in G \subset \mathbb{R}^n$$

$$f \mid_{\partial G} = 0$$

Condition (11) shows that most of the theory has been developed for PDE with homogeneous boundary conditions. In functional analysis terminology this is equivalent to say that $f$ belongs to a Sobolev space with compact support. In developing applications, some suitable procedure must be designed to transform the original boundary value problem with the solution function belonging to a space with non-compact support, which is the usual case, into an equivalent one with the solution function $f$ belonging to a space with compact support. Now using the theory of distributions and weak derivatives, it is possible to produce a variational formulation of eqn. (10) (see Oden and Reddy, 1976):
\((\nabla f, \nabla \phi)_0 = - (g, \phi)_0 \quad \phi \in C^\infty_0(G)\) \quad (12)

A series of theorems (see Showalter, 1977; Sawaragi et al., 1978; Griffel, 1981) converging to a major one state the existence and uniqueness of solution to problem \((10)\). The demonstrations for the Dirichlet and Neumann boundary conditions often use spatial discretization, thus providing a justification and an origin to approximation schemes often used in engineering problems (Raleigh-Ritz Galerkin method, method of moments, finite differences, finite elements, etc.). The theorem basically states that \(f \in H^1_0(G)\). Now some conclusions concerning the degree of smoothness of the solution to the system can be made.

Having in mind the concept of distributions, we see that it is sometimes possible to give a sensible interpretation to the derivatives of a function which is not smooth in the conventional sense.

A function \(f\) in \(L^2(G)\), for every closed and bounded \(G \subset \mathbb{R}^n\), is said to have a \(k\)th "weak" derivative if there is a \(g \in L^2_0(G)\) such that:

\[
\int_G g \phi \, dx = (-1)^{|k|} \int_G f \cdot \nabla^k \phi \, dx
\]

for all \(\phi \in C^\infty_0(G)\); \(g\) is called the \(k\)th weak derivative of \(f\), and we write \(\nabla^k f = g\).

We can extend the above concepts of first-order Sobolev space to the more general case of \(m\)-order Sobolev space. Let \(m\) be a non-negative integer. Denote by \(H^m(G)\) the set of functions \(f\) such that for \(0 \leq |\alpha| \leq m\) all weak derivatives \(D^\alpha f\) exist and are in \(L^2\) and equip \(H^m\) with an inner product and norm as follows:

\[
(f,g)_m = \sum_{|k| \leq m} \int_G D^k f \cdot D^k g \, dx
\]

\[
\|f\|_m^2 = (f,f)_m = \sum_{|k| \leq m} \int_G |D^k f|^2 \, dx
\]

\(H^m\) is again a Sobolev space of order \(m\). \(H^m\) is a proper subset of the set of functions with \(m\)th weak derivatives, for \(D^\alpha f\) are required to be in \(L\). Evidently \(H^0 = L^2\) and \((\cdot, \cdot)_0\), \(\| \cdot \|\) are the inner product and the norm of \(L^2\). Also note the relation:

\[C^\infty_0 \subset \ldots \subset H^{m+1}_0 \subset H^m \subset \ldots \subset H^0_0 = L^2\]

Now \(H^m\) is a Hilbert space (see Hutson and Pym, 1980, for proof).

Let \(H^m_0\) be the closure in \(H^m\) of \(C^\infty_0\). This is the \(m\)th order equivalent to the first-order Sobolev space defined above. As before there is a chain of inclusions:

\[C^\infty_0 \subset \ldots \subset H^{n+1}_0 \subset H^n_0 \subset \ldots \subset H^0_0 = L^2\]

Since functions \(C^\infty_0\) vanish near the boundary of \(G\), functions in their closure \(H^m_0\) may be expected to behave at the boundary in a manner which reflects that fact.
3. ABSTRACT EVOLUTION EQUATIONS

Most linear systems can be treated as abstract equations, which may be classified into two main categories: Those of the forms:

\[ Au = g \quad \text{in } G \]  \hspace{1cm} (16)

as in Section 2, and those of the forms:

\[ u' + Au = g \quad \text{in } G \]  \hspace{1cm} (17)

which are called evolution equations.

In the first type we can define the domain of the operator \( A \) as \( \text{Domain}(A) = \{ u \in U \text{ and } u \text{ satisfies the boundary conditions} \} \), where \( U \) is a Banach space. Hence in order to determine the solution of the abstract equation:

\[ Au = g \quad \text{on } U \]  \hspace{1cm} (18)

for all \( g \in U \), it is necessary to know whether \( A \) is invertible, which requires that \( A^{-1} \) be bounded and \( A \) be closed. Hence the problem of finding solutions consists in choosing appropriate spaces so that \( A \) is closed and then determine the conditions for the existence of an inverse. If \( A \) is in \( L^2(G) \) and \( \text{Domain}(A) = H^2(G) \cap H^1_0(G) \), then the problem would be to find \( u \in \text{Domain}(A) \), such that \( Au = g \), for \( g \in L^2(G) \).

Now let us consider the abstract evolution equation:

\[ u'(t) = Au(t) \quad u(0) = u_0 \in \text{Domain}(A) \]  \hspace{1cm} (19)

A solution in \([0, T]\) implies that \( u(t) \in \text{Domain}(A) \) for all \( t \in [0, T] \), \( u' \) exists, \( u'(t) = Au(t) \), and for each \( u_0 \in \text{Domain}(A) \) \( u(t) \) is unique.

Let \( J_t \) be the operator which assigns to each \( u_0 \in \text{Domain}(A) \) the value \( u(t) \) at time \( t > 0 \). Thus:

\[ u(t) = J_t u_0 \]  \hspace{1cm} (20)

\( J_t \in l[\text{Domain}(A)] \) for all \( t > 0 \). If we assume \( \text{Domain}(A) \) is dense in \( U \), it is possible to extend \( J_t \) to all \( U \). Now the solution to the problem at time \( t + s \) with \( u_0 \in \text{Domain}(A) \) is (Curtain and Pritchard, 1977, 1978):

\[ u(t + s) = J_{t+s} u_0 \]  \hspace{1cm} (21)

From the uniqueness of the solution:

\[ u(t + s) = J_t u(s) = J_t J_s u_0 \]  \hspace{1cm} (22)

Thus \( J_{t+s} u_0 = J_t J_s u_0 \) for \( u_0 \in \text{Domain}(A) \), and if \( \text{Domain}(\bar{A}) = U \):

\[ J_{t+s} = J_t J_s \]  \hspace{1cm} (23)

An operator with these properties is called a semigroup (Curtain and Pritchard, 1977).

If \( J_t \) is an operator on \( \mathbb{R}^+ \) to \( l(U) \), where \( U \) is a Banach space, and satisfies:

\( J_{t+s} = J_t J_s \quad t \geq 0; \quad J_0 = I \), the identity operator; and (3)
\[ \|J_t u - u\|_U \to 0 \text{ as } t \to 0 \text{ for all } u \in U; \text{ then } J_t \text{ is said to be a strongly continuous semigroup.} \]

Now consider the inhomogeneous evolution equation:

\[ u' + Au = g; \quad u(0) = u_0 \in \text{Domain}(A) \tag{24} \]

If \( g \in C[0, T; U] \) and if there is a solution for eqn. (24) on \([0, T]\), then:

\[ u(t) = J_t u_0 + \int_0^t J_{t-s} g(s) ds \tag{25} \]

We can expand the semigroup theory developed for operators in Banach spaces to partial differential operators in Hilbert spaces.

Many initial-boundary value problems for partial differential equations can be regarded as abstract evolution equations in appropriate function spaces. In particular, the equations describing groundwater flow (i.e. the unsteady flow in confined or unconfined aquifers) may be reduced to an evolution equation. The evolutional equation approach has two obvious advantages in treating PDE: The conceptual simplification and the notational economy of a wide range of problems.

Consider an evolution problem: Find a \( u \in H^m_0(G) \) such that:

\[ \frac{\partial u(t)}{\partial t} + Au(t) = g \tag{26} \]
\[ u(x, 0) = u_0(x) \tag{27} \]

where \( u_0 \) and \( g \) are given in \( H \). Sawaragi et al. (1978) have proved the existence and uniqueness of the solution of eqns. (26) and (27). We will seek its abstract analogue when we study stochastic evolution equations in Part II.

For the homogeneous case with \( g = 0 \) the solution is:

\[ u(t) = J_t u_0 \tag{28} \]

where \( J_t \in l(H, H) \) is an evolution operator in the Hilbert space associated with \( A \). Then we can represent the solution of eqns. (26) and (27) in the following form:

\[ u(t) = J_t u_0 + \int_0^t J_{t-s} g(s) ds \tag{29} \]

For the properties of the evolution operator \( J_t \) and a good introduction of the extensive body of the theory on evolution equations see Ladas and Lakshmikantham (1972). See also Butzer and Berens (1967) for an in depth treatment of semigroup theory.

4. APPLICATION TO ONE-DIMENSIONAL GROUNDWATER FLOW

As an example of the use of functional analysis and semigroups, consider the one-dimensional groundwater flow in a homogeneous phreatic aquifer. The governing equation is given by (see Bear, 1972, 1979):
where \( h \) is the hydraulic head (m); \( x \) is the horizontal coordinate (m); \( t \) is the time coordinate (day); \( S \) is the aquifer storage coefficient or specific yield; \( K \) is the aquifer hydraulic conductivity (m per day); \( I \) is the input function representing deep percolation to the aquifer and assumed uniformly distributed along \( x \) (m per day) (see Fig. 1). General assumptions of homogeneity, essentially horizontal flow, constant hydraulic conductivity and constant hydraulic gradient over the whole depth of the aquifer have been used in the development of eqn. (30), called the Boussinesq equation. After a simple linearization procedure (Bear, 1979), eqn. (30) reduces to:

\[
\frac{\partial}{\partial x} \left( h \frac{\partial h}{\partial x} \right) = \frac{S}{K} \frac{\partial h}{\partial t} - \frac{L}{K}
\]

where \( T \) is the average constant transmissivity and \( h \in H^1(0, L) \). Applying the boundary conditions:

\[
h(0, t) = C \quad 0 < t
\]

\[
\frac{\partial h}{\partial x} (L, t) = 0 \quad 0 < t
\]

and:

\[
\frac{\partial h}{\partial x} (x', 0) = \frac{dh_0}{dx} (x) \quad 0 < x < L
\]

Note that the initial condition (34) is defined in terms of the gradient of the initial water table.

As stated before, it is necessary to transform our functional space into an equivalent one with compact support in order to treat the system equation as an abstract evolution equation in a Sobolev space. By defining:

\[
y(x, t) = h(x, t) - V(x)
\]
where \( V(x) \) is a sufficiently smooth known function satisfying the steady state solution given by (Serrano, 1985):

\[
V(x) = \frac{I}{T} \left( Lx - \frac{x^2}{2} \right) + C
\]  

(36)

it is possible to transform the system of eqns. (31)–(34) in \( h \in H^1(0, L) \) into an equivalent one in \( y \in H^1_0(0, L) \):

\[
\frac{\partial y}{\partial t} - \frac{T}{S} \frac{\partial^2 y}{\partial x^2} = 0
\]  

(37)

Subject to:

\[
y(0, t) = 0
\]  

(38)

\[
\frac{\partial y}{\partial x} (L, t) = 0
\]  

(39)

\[
y(x, 0) = h_0(x) - V(x) = y_0(x)
\]  

(40)

Now we can treat eqn. (37) as an abstract evolution equation in the Sobolev space \( H^1_0(0, L) \) with the operator \( A \) given by \( Ay = T/S \left( \partial^2 y / \partial x^2 \right) \). It is an operator on the Hilbert space \( H^0 \), with:

\[
\text{Domain}(A) = \left\{ y \in H^0; \quad \frac{\partial y}{\partial x}, \frac{T}{S} \frac{\partial^2 y}{\partial x^2} \in H; \quad y(0, t) = 0; \quad \frac{\partial y}{\partial x} (L, t) = 0 \right\}
\]

At this point we can use the theorems of functional analysis to state that there exists a distributional solution satisfying the problem in eqns. (37)–(40). The distribution satisfying eqns. (37)–(40) is a unique solution and belongs to the Sobolev space \( H^1_0(0, L) \), that is, the distribution has a well-defined first weak derivative among other properties. Since the problem is a deterministic one, we can also say that to the unique distributional solution corresponds a unique functional solution in the classical sense. The above observations and conclusions become particularly important if the problem in eqns. (31)–(34) had not been solved before, which would be the case if the system were randomly excited or were subject to random boundary conditions.

Let us consider the physical nature of the problem in Fig. 1, which represents a half-cross section of a symmetric watershed bounded by the river channel at \( x = 0 \) and by the groundwater divide at \( x = L \). Only for gently sloping watersheds possessing the above characteristics can the eqn. (31) be considered valid. As shown in the sketch, the water level at \( x = 0 \) is assumed equal to \( C \) (m), a constant. Physically this would mean that the river stage is controlled, such as in an irrigation district or a semiurbanized area.

The boundary at \( x = L \) is easier to justify, because in a homogeneous aquifer the groundwater divide coincides with the topographic divide. The
assumption of horizontal flow and horizontal impermeable base is only acceptable in the case of gently sloping watersheds.

The initial water-table condition \( h_0 \) may be represented by a quadratic or a cubic function of \( x \) fitted to piezometric level measurements available at a few points across the watershed. In Part II we will study a stochastic procedure for modeling the initial condition.

The input function may be determined by the use of any deterministic model such as the Holtan infiltration model (Viessman et al., 1977) in combination with some evaporation and water consumption estimates. We will assume that the input function \( I \) is constant.

It is seen that \( A \) generates a semigroup \( J_t \) given by (see Serrano, 1985):

\[
J_t y = \sum_{n=1}^{\infty} \frac{L}{\lambda_n^2} \exp \left( -\frac{\lambda_n^2 Tt}{S} \right) \sin(\lambda_n x) \int_{0}^{L} y(s) \sin(\lambda_n s) ds
\]

where the eigenvalues \( \lambda_n \) are given by (Powers, 1979):

\[
\lambda_n = \left( \frac{2n - 1}{2L} \right) \pi, \quad n = 1, 2, 3, \ldots
\]

Hence the solution to eqns. (37)–(40) is given by eqn. (28) as:

\[
y(x, t) = J_t y_0
\]

Hence the solution to our original problem in eqns. (31)–(34) is:

\[
h(x, t) = \frac{T}{I} \left[ Lx - \frac{x^2}{2} \right] + C + \sum_{n=1}^{\infty} \sin(\lambda_n x) \cdot \exp \left( -\frac{\lambda_n^2 Tt}{S} \right) \cdot \left\{ \frac{2}{L} \int_{0}^{L} [h_0 - V(x)] \cdot \sin(\lambda_n x) dx \right\}
\]

Equation (44) allows us to predict the water table with time for a given input \( I \) and initial condition \( h_0 \). A similar solution has been obtained by Venetis (1969, 1971) for studies on the recession of unconfined aquifers and estimation of infiltration. We will study the stochastic analogue of eqn. (44) and will see how randomly defined initial conditions and/or forcing function term enter the equation and affect the properties of the output function.

Difficulties in defining \( h_0 \) are well known in hydrologic practice, since the position of the water table for a particular point depends on a number of factors. The three more important factors affecting the response of the water table to climatic changes are the depth of the water table below the ground surface, soil type and microtopography (Ward, 1963). The magnitude of the response of the water table to rainfall and evapotranspiration will tend to vary inversely with the distance between the ground surface and the water table. Thus in given climatic conditions both the daily and seasonal...
fluctuations in water table height will be more marked in areas where the water table is close to the ground surface than where it is found at greater depths (Ward, 1963).

The effect of soil type is such that the water-table fluctuations will be greater in clay than in gravel. The greater percentage of pore space in clays will result in a larger rise in water table for a given rainfall depth, than for sands. Also higher permeability in sands will make the water drain faster and dispersed. Furthermore, capillary rise is more important in clays and the loss of water by evapotranspiration is higher in clays (Ward, 1963).

The accumulation of surface and subsurface runoff in the depressions after heavy rainfall ensures higher soil moisture content in these areas, thereby promoting higher percolation and higher water-table response in these areas than in higher areas of the flood plain. In a depression, when the water table is at ground surface, any further rainfall will be carried away as surface runoff along the depression bottom thereby preventing a rise in water-table level. When the water table is falling elsewhere in the floodplain, the drainage of groundwater from the surrounding higher areas will tend to maintain water-table height in the depression at ground surface level. Finally, water table in gravel drains faster and water table in clays rises faster (Ward, 1963).

Obviously the model eqn. (44) does not predict the local variations in the water table nor does it take into consideration the factors affecting the water-table shape discussed above. As will be seen later in this section, the model considers the local initial variations of the water table in the production of transient profiles for short periods of time after initiation, but soon the water table will invariably take a smooth shape.

The above discussion indicates that there is a need for research concerning the functional relationship between the ground surface elevation and the groundwater table elevation. It also seems that a better approach in modeling the free-surface profile is provided by representing it as a stochastic process accounting for the uncertainties in the determination of this relationship (Part II).

To obtain some simple versions of the solution eqn. (44), use will be made of a well-known observation that the water-table profile approximately follows the profile of the ground surface (Toth, 1962, 1963). Use will also be made of Toth's approach, who simulated the effect of soil depressions on the water table as a sine wave superimposed on a straight line, whose slope was similar to the average slope of the ground surface. Toth used several different cycles and derived the solution to Laplace's equation to draw some conclusions about regional groundwater flow.

Let us first derive the exact expression for the integral of eqn. (44) for a specific representation of the initial condition. The initial condition is given by:

\[ h_0 = C + Dx + A'\sin(b'x) \] (45)
and:
\[ D = \tan(\alpha) \]  \hspace{1cm} (46)
\[ A' = \frac{A}{\cos(\alpha)} \]  \hspace{1cm} (47)
\[ b' = \frac{b}{\cos(\alpha)} \]  \hspace{1cm} (48)

where \( \alpha \) is the average ground slope, \( A \) is the amplitude of the sine wave (m) and \( b \) is the frequency given by:

\[ b = \frac{\pi N}{L} \]  \hspace{1cm} (49)

with \( N \) the number of half cycles. Toth (1963) found that for \( \alpha \leq 3^0 \) this representation produces satisfactory results. Using eqn. (36):

\[ [h_0 - V(x)] = [C + Dx + A'\sin(bx)] - \left[ \frac{I}{T} \left( Lx - \frac{x^2}{2} \right) + C \right] \]  \hspace{1cm} (50)

The Fourier coefficient is derived as:

\[ b_n = \frac{2}{L} \left[ \left( D - \frac{I}{T} L \right) \int_0^L x \sin(\lambda_n x) dx + \frac{I}{2T} \int_0^L x^2 \sin(\lambda_n x) dx \right. \]

\[ + A' \int_0^L \sin(b'x) \sin(\lambda_n x) dx \]  \hspace{1cm} (51)

Solving the integrals we obtain:

\[ b_n = \frac{2}{\lambda_n^2 L} \left[ \left( D - \frac{I}{T} L \right) (-1)^{n-1} + \frac{I}{T} \left( L(-1)^{n-1} - \frac{1}{\lambda_n} \right) \right] \]

\[ + \frac{A'}{L} \left[ \frac{\sin(b' - \lambda_n) L}{(b' - \lambda_n)} - \frac{\sin(b' + \lambda_n) L}{(b' + \lambda_n)} \right] \]  \hspace{1cm} (52)

Note that the term in the braces corresponds to the coefficient due to the linear initial condition, and the following term in the square brackets corresponds to the coefficient due to the added sinusoidal condition. Computations for various values of \( N \) were carried out up to an accuracy of 0.01 m, which was considered to be an acceptable approximation.

Several runs were made to observe the behaviour of the output variable \( h(x, t) \). For the purposes of the present study the value of the hydraulic conductivity was chosen to be equal to \( 10^{-5} \) m s\(^{-1}\), which corresponds to an average soil of silty sand. The specific yield \( S \) varies between 0.01 and 0.30 and an average value of 0.14 was chosen for the tests. The value of \( N \) in the
initial condition $h_0(x)$ was also adjusted so that $d h_0 / dx (L)$ becomes zero and the boundary condition is not violated. Approximate values of $N$ for which this condition is preserved were found to be $N = 1.09, 2.64, 3.39, 4.57, 5.43, \ldots$. Figure 2 shows the transient profiles of the water table in the absence of deep percolation, for the case of $\alpha = 2^0$ and a 1.09 half-cycles sine wave initial condition. Figures 3 and 4 show the transient free-surface profiles for $N = 2.64$ and $N = 3.39$, respectively. Note that the initial shape of the free-surface profile changes in less than a month towards a smoother curve. This will contradict the effect of the local conditions on the water-table shape discussed before. Another interesting feature is that the shape of the water table tends to be the same after prolonged periods of time and similar $\alpha$, especially when the initial condition curve is symmetric with respect to the straight line representing the average ground-surface slope.

Figure 5 shows the profiles for a constant deep percolation of 0.1 mm h$^{-1}$ and $N = 4.57$. The assumption of constant deep percolation throughout the period of simulation is not an unusual fact during the spring snowmelt or
during rainy months. The value was taken from the results of an extensive numerical simulation performed in the Middle Thames watershed in southern Ontario (Canada) by the use of the Holtan infiltration model for a variety of soil parameters (Serrano, 1982). The tests in the above study were conducted in order to clarify the relative importance of subsurface flow for different kinds of time distribution of rainfall intensity. The resulting subsurface flow supply was almost uniformly distributed over the entire period of the storm, which included about five days from the initiation of the storm. The maximum hourly rate of subsurface flow supply was about 0.4 mm h$^{-1}$, regardless of the input rainfall intensity (Serrano, 1982).

The above simple example has been used as an illustration of the concepts of functional analysis, evolution equations and semigroups of operators in Sobolev spaces. The concepts will be used in the development of the theory of stochastic PDE in Part II and the example itself will be used for comparison with the equivalent stochastic groundwater flow problem.
5. CONCLUDING REMARKS

Functional analysis appears to be an excellent and rigorous approach to the analysis of deterministic partial differential equations in groundwater flow. The authors believe these concepts will find with time increasing acceptance among water resources researchers. The flexibility of the theory allows one to obtain solutions to a variety of problems by treating the equations as abstract evolution equations for elliptic partial differential operators in an appropriate Sobolev space.

The use of semigroup theory presents promising results in finding solution expressions of abstract evolution equations. The difficulties in deriving the semigroup for a particular partial differential operator, which can be found by initially using traditional methods, are compensated by the economy and flexibility gained in finding solutions for the same system subject to different initial conditions, different boundary conditions or different forcing function. Furthermore, properties of semigroups give additional advantages when dealing with the equivalent stochastic problem, as will be seen in Part II, thus giving a unified conception of the deterministic and the stochastic problem.

The example presented is inspired in the early work by Toth (1963), except that the Boussinesq equation is used instead of Laplace. His representation of the top-side boundary condition in the aquifer as a sine wave is used to obtain an expression for the free-surface boundary condition. Changing the number of cycles seems to be an adequate representation for the water table following surface depressions. A random equivalent of these expression will prove most useful in treating random free-surface profiles in Part II. Although the potential applications of this model are good, there is a need for further research concerning the functional relationship between the ground level and the water-table level.

LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>An interval limit, a constant</td>
</tr>
<tr>
<td>$A$</td>
<td>A partial differential operator</td>
</tr>
<tr>
<td>$A^*$</td>
<td>Dual operator of $A$</td>
</tr>
<tr>
<td>b</td>
<td>An interval limit, a constant</td>
</tr>
<tr>
<td>$C$</td>
<td>Constant piezometric head at $x = 0$, space of continuous linear functionals</td>
</tr>
<tr>
<td>$C[a, b]$</td>
<td>Space of continuous linear functionals in the closed interval $[a, b]$</td>
</tr>
<tr>
<td>$C_0^\infty$</td>
<td>Space of infinitely differentiable distributions with compact support</td>
</tr>
<tr>
<td>$d^n$</td>
<td>$n$th ordinary derivative</td>
</tr>
<tr>
<td>$D$</td>
<td>A constant</td>
</tr>
<tr>
<td>$D^k$</td>
<td>$k$th partial weak derivative</td>
</tr>
<tr>
<td>$f$</td>
<td>A functional, a distribution</td>
</tr>
<tr>
<td>$</td>
<td>f</td>
</tr>
<tr>
<td>$|f|$</td>
<td>Norm of $f$</td>
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<tr>
<td>$|f|_0$</td>
<td>&quot;Zero&quot; norm or $L_2$-norm of $f$</td>
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\[ \|f\|_1 \] \begin{align*} &\text{"One" norm of } f \text{ or norm in the sense of } H^1 \end{align*} 
\langle f, \phi \rangle \begin{align*} &\text{Inner product of functional } f \text{ with a test function } \phi \end{align*} 
g \begin{align*} &\text{Forcing function, generally} \end{align*} 
G \begin{align*} &\text{An } n \text{th dimensional domain} \end{align*} 
\bar{G} \begin{align*} &\text{The closure of } G \end{align*} 
\partial G \begin{align*} &\text{The boundary of the } G \text{ domain} \end{align*} 
h \begin{align*} &\text{Piezometric head, a function} \end{align*} 
\bar{h} \begin{align*} &\text{Mean groundwater depth} \end{align*} 
h_0 \begin{align*} &\text{Initial piezometric head} \end{align*} 
H^m \begin{align*} &\text{An } m \text{th order Sobolev space} \end{align*} 
H^1_0 \begin{align*} &\text{First-order Sobolev space of first differentiable distributions with compact support} \end{align*} 
H^1(G) \begin{align*} &\text{First-order Sobolev space of all functions } f \in L^2(\bar{G}) \text{ such that all the first partial derivatives belong to } L^2(\bar{G}) \end{align*} 
I \begin{align*} &\text{Source term, a function} \end{align*} 
J \begin{align*} &\text{Analytic semigroup} \end{align*} 
k \begin{align*} &\text{Derivative order, an index} \end{align*} 
K \begin{align*} &\text{Hydraulic conductivity} \end{align*} 
l \begin{align*} &\text{Derivative order, linear space} \end{align*} 
L \begin{align*} &\text{Aquifer length, linear space} \end{align*} 
L_\infty \begin{align*} &\text{Space of square-integrable distributions} \end{align*} 
n \begin{align*} &\text{Counter, normal direction} \end{align*} 
N \begin{align*} &\text{A constant, number of half cycles for the sinusoidal representation of water table} \end{align*} 
s \begin{align*} &\text{A dummy variable} \end{align*} 
S \begin{align*} &\text{Specific storage} \end{align*} 
t \begin{align*} &\text{Time coordinate} \end{align*} 
T \begin{align*} &\text{Aquifer transmissivity} \end{align*} 
u_0 \begin{align*} &\text{Initial condition of the system} \end{align*} 
u(t) \begin{align*} &\text{State of the system at time } t \end{align*} 
U \begin{align*} &\text{A Banach space} \end{align*} 
V(x) \begin{align*} &\text{Steady-state function} \end{align*} 
w \begin{align*} &\text{"Weakly"} \end{align*} 
W^m_p \begin{align*} &m\text{th order Sobolev space, where } p \text{ is the power of the space norm and } m \text{ is the order of the derivative inside the norm} \end{align*} 
x \begin{align*} &\text{Spatial coordinate} \end{align*} 
X \begin{align*} &n\text{th dimensional vector space of } x \end{align*} 
y \begin{align*} &\text{A function} \end{align*} 
\alpha \begin{align*} &\text{Mean water table slope} \end{align*} 
\lambda_n \begin{align*} &\text{Eigenvalues} \end{align*} 
\pi \begin{align*} &3.1415 \ldots \end{align*} 
\phi \begin{align*} &\text{A "smooth" function} \end{align*} 
\Sigma \begin{align*} &\text{Summation} \end{align*} 
\nabla \begin{align*} &\text{Gradient} \end{align*} 
\partial^n \begin{align*} &n\text{th partial derivative} \end{align*} 
\mathbb{R} \begin{align*} &\text{Real space} \end{align*} 
\mathbb{R}^n \begin{align*} &n\text{th dimensional real space} \end{align*} 
\infty \begin{align*} &\text{Infinity} \end{align*} 
\subset \begin{align*} &\text{Subset of} \end{align*} 
\cap \begin{align*} &\text{Intersection} \end{align*}
REFERENCES