Semigroup solutions to stochastic unsteady groundwater flow subject to random parameters

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Abstract: Two methods for the solution of partial differential equations (PDE) for the general case of random in time physical parameters are presented and their application to the solution of unsteady regional groundwater flow equations are illustrated. The first method is the semigroup approach which directly offers a solution without resorting to “closure approximations” (hierarchy techniques), perturbation techniques, or Montecarlo simulation techniques. The semigroup approach can also handle the general stochastic problem when randomness also appears as initial conditions, boundary conditions or forcing terms. The second method is an approximation scheme to obtain the semigroup solution in complex cases and permits the solution of equations with more than one random coefficient.

Key words: Stochastic groundwater flow, semigroups, random in time Gaussian process, random physical parameters, stochastic partial differential equations.

1 Introduction
The present paper develops a methodology for the solution of partial differential equations (PDE) of unsteady groundwater flow when the parameters are defined as stochastic processes. The main application considered is the one-dimensional transient regional groundwater flow subject to random transmissivity, but the procedure may be used to solve more complex higher-dimensional equations for groundwater flow and contaminant transport.

One of the difficult problems in modeling groundwater flow systems is the quantitative determination of parameters such as transmissivity and specific storage. It is important to develop an appropriate mathematical description of the physical processes governing porous media flow, but it is equally important to measure and estimate aquifer parameters which best describe field conditions. Several laboratory and field methods have been developed for in situ measurement and detailed sample evaluation of hydraulic conductivity. These methods have proven useful in many field situations where stratigraphic analysis of the aquifer system makes it reasonable to assume homogeneity and isotropy of the porous media (see for example Bear 1979; Freeze and Cherry 1979).

When homogeneity is assumed, but isotropy may not be the case, the hydraulic conductivity $K$ and therefore transmissivity $T$, is defined as a second rank tensor whose components completely describe its magnitude. This implies that the hydrologist must estimate six distinct components for a three-dimensional model and
three such components for a simplified two-dimensional model. Since there is considerable work, uncertainty, cost and difficulty in determining hydraulic conductivity even for a homogeneous isotropic porous media, the tensorial description of hydraulic conductivity is not a practical one.

If the aquifer is classified as inhomogeneous, then it is necessary to know the spatial functional relationship of each component. To this we have to add the complexities derived from the inadequate description of dynamic boundary conditions and recharge.

Fortunately, field situations often allow simplifications of the general model: Many shallow alluvial aquifers and layered systems may be individually modeled as one-dimensional flow in homogeneous porous media; two-dimensional simplifications, linearizations and numerical approximations, based on the concept that hydraulic conductivity is almost constant at specified blocks, are now widely used in hydrologic practice (see for example, Pinder and Gray 1977).

Stochastic analysis of groundwater flow has emerged as a tool to evaluate and consider the uncertainties derived from the above description and the errors originated in model simplifications and measurement. Most stochastic approaches in the recent past have concentrated on a conceptual elaboration of the porous media system as a space and time stochastic entity (Dagan 1982 1981 1979; Delhomme 1979; Dettinger and Wilson 1981; Freeze 1975; Gelhar 1977 1974; Gutjahr and Gelhar 1981; Kottegoda and Katsuik 1983; Naff and Vecchia 1986; Sagar 1978; Tang and Pinder 1979; Townley and Wilson 1985). Among the methods of analysis used are stochastic continuum theory, stochastic Green's functions, random walk models, spectral analysis, perturbation solutions for small randomness, and Monte carlo simulations. These approaches are useful when the covariance structure of the hydraulic conductivity is known a priori. However the task of estimation and field measurement of the spatial statistical properties of K is too enormous to be practical.

An alternative stochastic approach considers the governing equation subject to time stochastic inputs, random initial conditions and/or random boundary conditions (Serrano et al. 1985 a, b, c; Sagar 1979). Uncertainty and errors generated by model assumptions and approximations, and errors in measurement of parameters, boundary conditions are considered by solving the resulting stochastic PDE. The solution of stochastic PDE has become possible because of recent developments in the functional analysis of abstract stochastic evolution equations and in analytic semigroups of operator equations (see Sawaragi et al. 1978; Curtain and Pritchard 1977; Curtain 1978). Specific field applications of these models have shown that this procedure enhances the prediction capabilities of equivalent deterministic models by a straight forward extension to consider uncertainty terms, and uses features of deterministic analytical and numerical procedures available in the literature (Serrano and Unny 1986).

The present paper is evolved from the work by Serrano, Unny and Lennox, (1985a, b and c) who presented a general functional-analytic theory for the analysis and solution of stochastic PDE. Although in the above work it was implied that the theory would be expandable to the more complex case of random operator equations, no specific details were given to cover these type of problems. This article attempts to present such a methodology. In Section 2 we introduce the general procedure to obtain the random semigroup solution of an evolution equation when the partial differential operator is stochastic. To avoid unnecessary repetition, the reader is referred to Serrano, Unny and Lennox (1985a b c) for the definitions and the concepts of functional analysis and semigroups of random PDE's in groundwater hydrology. In Section 3 an alternative procedure is presented to approximate and successively improve the calculation of the semigroup solution for this type of
equation. In Section 4 we illustrate the application of both approaches to the solution of the one-dimensional groundwater flow equation with random transmissivity. The stochastic properties of groundwater levels with time are studied.

2 The semigroup solution of a random operator equation

Let us consider the system governed by a general linear stochastic evolution equation of the form

\[ \frac{\partial u}{\partial t} + A(x, t, \omega)u = g(x, t, \omega), \quad (x, t, \omega) \in G \times [0, T] \times \Omega, \]

\[ Q(x, t, \omega)u = F(\omega), \quad (x, t, \omega) \in \partial G \times [0, T] \times \Omega, \]

\[ u(x, 0, \omega) = u_0(x, \omega), \quad x \in G \times \Omega \]

where \( u(x, t, \omega) \in H^m(G) \times \Omega \) is the system output; \( g \in L^2(\Omega, B, P) \) is a second-order random function representing the forcing term; \( G \subset \mathbb{R}^3 \) is an open domain subset of the three-dimensional real space with boundary \( \partial G \); \( t \) is the time coordinate; \( 0 < T < \infty \); \( Q \) is the boundary operator; \( x \) represents three-dimensional spatial domain; \( A \) is a formal random partial differential operator given by

\[ Au = \sum_{|k| + |l| \leq m} (-1)^{|k|} D^k(p_{kl}(x, t, \omega))D^l u. \]

(2)

Here, \( H^m \) is the \( m \)-th order Sobolev space of second-order random functions; \( \Omega \) is the basic probability sample space of elements \( \omega \); \( L^2(\Omega, B, P) \) is the complete probability space of second-order random functions with the probability measure \( P \) and \( B \) the Borel field of class of \( \omega \) sets; \( D \) is differentiation in Hilbert space; and \( p_{kl}(x, t, \omega) \) in this case are randomly varying stochastic processes representing the system parameters, which are assumed bounded and mean-squared continuous on \( [0, T] \). These stochastic parameters can be represented as \( p_{kl} = E\{p_{kl}(x, t)\} + p_{kl}^r(t, \omega) \), where \( E\{\cdot\} \) is the expectation operator, which we assume exists. Any deterministically observable trend and/or periodicity in the parameters is included in the term defined by the expectation.

Randomness may enter the system in several ways: (i) The random initial value problem, when \( u_0 \) is random. (ii) The random boundary value problem, when \( F \) is random. (iii) The random forcing problem, when \( g \) is random. (iv) The random operator problem, when \( A \) or \( Q \) is random. (v) The random geometry problem.

Serrano, Unny and Lennox (1985a, b, c) presented a general functional-analytical theory of stochastic PDE in groundwater flow and gave specific examples for the application of the theory to solve groundwater problems for cases (i), (ii) and (iii) above. Our intention here is to show the extension of the theory to cover the more complex problem of random operator equations (case iv above).

The system of equations (1), where \( u \) belongs to the \( m \)-th order Sobolev space \( H^m(G) \), can be transformed into an equivalent system where the output process (solution process) \( u \) belongs to the \( m \)-th order Sobolev space with compact support \( H^m_0(G) \). This space will contain random distributions vanishing at the boundary. In classical mathematics terminology, we will have the equivalent stochastic PDE with homogeneous boundary conditions. The transformed system equation can be written as

\[ \frac{\partial u}{\partial t} + A(x, t, \omega)u = g(x, t, \omega), \quad u \mid_{\partial G} = 0. \]

(3)
\( u(x, 0) = u_0 \)

where \( g \) now contains information on the boundary conditions.

We can describe the solution process \( u(x, t, \omega) \) as

\[
\begin{align*}
u &= U(t, 0, \omega)u_0 + \int_0^t U(t, s, \omega)g(s, \omega)ds.
\end{align*}
\]

(4)

We call \( U(t, s, \omega) \) the random evolution operator associated with \( A(x, t, \omega) \), and \( ||U(t, s, \omega)|| \leq 1 \) for \( t \geq s \), where \( ||.|| \) is the norm of the \( H_0^0 \) space. Because \( A \) is dependent on \( t \) and \( \omega \) in addition to \( x \), \( U \) will generally depend on \( x, t \) and \( \omega \) and the derivation of \( U \) will be a difficult task. Hence efforts should be made to manipulate the PDE to transform it into one in which the operator \( A \) is not time dependent. This can be done by transforming the trend and/or the periodic component in the parameters \( p_{kl} \) in Eq. (2), as is routinely done in the analysis of random signals, and leaving for study only the stationary case. Hence, ideally \( E\{p_{kl}\} \) is a constant and \( p_{kl}^\prime \) is a zero mean random process of finite variance. If \( A \) is time independent, then we have

\[
\begin{align*}
U(t, s, \omega) &= J(t-s, \omega).
\end{align*}
\]

(5)

The function \( t \rightarrow J_t \), which is defined for all \( t \geq 0 \), is then a strongly continuous semigroup in \( H \). Semigroup theory is rapidly becoming an important branch in mathematics because of the increasing number of applications. For example, complex one-dimensional systems of ordinary differential equations can be reduced to simpler higher-dimensional problems by the use of semigroup theory. Semigroups can be used in Markov processes and ergodic theory, and non-linear operators theory. See, for example, Goldstain (1985) for a very interesting review of the semigroup theory. In terms of the semigroup, the solution of Eq. (3) for a time-independent operator \( A \) is

\[
\begin{align*}
u &= J(t, \omega)u_0 + \int_0^t J(t, s, \omega)g(s, \omega)ds.
\end{align*}
\]

(6)

Theoretically we may be interested in finding the joint distribution function of all orders that characterize the process \( u \). This task is frequently complicated and in many situations provides more than what is required. We can often consider simpler and necessarily less complete characterizations in the form of expectations, dispersions, covariances, joint moments, correlations, etc., which are called statistical measures. This view is supported by the fact that it is usually not feasible to collect enough field information to evaluate the joint probability density function of the input process \( g(x, t, \omega) \) and the parameters \( p_{kl}(t, \omega) \). Therefore, from the practical point of view, it is possible to calculate only the first few low order moments of the solution process \( u \). The first two moments give considerable information about the joint probability density function of \( u \). The mean value of \( u \), assuming deterministic initial conditions is given by

\[
\begin{align*}
E\{u(x, t)\} &= E\{J(t, \omega)\}u_0 + \int_0^t E\{J(t-s, \omega)\}E\{g(x, s, \omega)\}ds
\end{align*}
\]

(7)

where statistical separation occurs between the semigroup and the forcing term. Physically this stems from the independent behaviour of the input function and the system parameters.

We immediately see that if we can derive the semigroup \( J \), this approach has obvious advantages over the usual methods of solution, namely hierarchy or averaging methods, perturbation methods ("small randomness size" assumptions), and
Monte Carlo simulation techniques.

Now the second moment of \( u \) in Eq. (6) is given by

\[
E \{ u(t_1)u(t_2) \} = E \{ [J_{t_1}u_0 + \int_0^{t_1} J_{t_1-s} g(s) ds] \cdot [J_{t_2}u_0 + \int_0^{t_2} J_{t_2-s} g(\xi) d\xi] \} 
\]

(8)

where \( \omega \) has been omitted for convenience. Assuming that \( g \) is a zero-mean stochastic process and solving we obtain

\[
E \{ u(t_1)u(t_2) \} = E \{ J_{t_1+t_2} u_0^2 \} + \int_0^{t_1+t_2} E \{ J_{t_1+t_2-s} \} E \{ g(s)g(\xi) \} dsd\xi.
\]

(9)

Note that it is not absolutely necessary to assume \( g \) is white Gaussian noise, but we do so for illustration purposes. A white Gaussian process in time and in space,

\( \{ g(x, t, \omega), x \in H, t \in T, \omega \in \Omega \} \),

is defined as (Sawaragi et al, 1978)

\[
E \{ g(x, t, \omega) \} = 0, \quad E \{ g(x_1, t_1)g(x_2, t_2) \} = q(x_1, t_1)\delta(x_1-x_2)\delta(t_1-t_2), \quad q > 0
\]

where \( q \) is the variance parameter. This is a special case. The general case for unsteady conditions is considered in the present article. It is assumed that \( g \) is a white Gaussian process in time defined by

\[
E \{ g(x, t, \omega) \} = 0, \quad E \{ g(x_1, t_1)g(x_2, t_2) \} = q(x_1, x_2, t_1)\delta(t_1-t_2), \quad q > 0.
\]

We shall study an application example in Section 4.

3 Approximate calculation of the semigroup solution

When the semigroup associated with the operator \( A \) in Eq. (3) is difficult to derive, the random component \( A \) enters in a complex way, or there is more than one random parameter, an approximate calculation method of the semigroup should be adopted. We follow in this section a procedure similar to the one used by Adomian (1970, 1971a, 1971b) to decompose an ordinary stochastic differential operator into an infinite series in order to approximate the corresponding stochastic Green's function. A similar procedure was used by Tang and Pinder (1979) to obtain a numerical solution of the advective-dispersive equation. We shall present a semigroup formulation for a general stochastic partial differential operator.

Let us write the operator \( A \) as

\[
A(x, t, \omega) = A(x, t) + R(t, \omega)
\]

(10)

where \( A(x, t) \) is the deterministic component and \( R(t, \omega) \) the random part or the portion of the partial differential operator containing the random parameters. Thus Eq. (3) can be written as

\[
\frac{\partial u}{\partial t} + Au = g - Ru,
\]

(11)

\[
u_{\partial \omega} = 0; \quad u(x, 0) = u_0,
\]

whose solution is given by Eq. (6) as

\[
u = J_t u_0 + \int_0^t J_{t-s} g(s) ds - \int_0^t J_{t-s} Ru(s) ds.
\]

(12)

It is not possible to solve Eq. (12) explicitly. Now we decompose \( u \) in the right hand side as an infinite series \( u = \sum_{i=1}^{\infty} u_i \). Equation (12) becomes
where the semigroup $J$ is now deterministic. Identifying $u_1$ as the preceding term, $\int_0^t J_{t-s} g(s) ds$, we can determine each $u_i$ in terms of the preceding $u_{i-1}$. Thus

$$u = J_t u_0 + \int_0^t J_{t-s} g(s) ds - \int_0^t \int_0^s R J_{t-s} R J_{t-s} \gamma g(\tau) d\tau ds - \int_0^t \int_0^s \int_0^\tau R J_{t-s} R J_{t-s} \gamma g(\xi) d\tau d\xi ds - \ldots$$

where the last term in the series contains $u$. The basic idea here is that a random semigroup operator, which may be difficult to derive in particular cases, can be determined in an easily computable series by decomposition of the differential operator $A(x, t, \omega)$ into a deterministic operator $A(x, t)$ whose semigroup is known or found with little effort, and a random operator $R(t, \omega)$ whose contribution to the total semigroup $J_t\omega$ can be found in series form.

The convergence question will not be treated here, since it has been discussed elsewhere (Adomian, 1970, 1971a, 1971b). It is enough to say that beyond the norm restriction, that is $\|J\| \leq 1$ which is useful in numerical solutions of the stochastic PDE, we can intuitively see that the $n$th integrand in Eq. (14) approaches zero because of the semigroup negative exponential with a power of an increasing number of eigenvalues.

The mean value of $u$ is obtained by truncating Eq. (14) and taking expectations:

$$E\{u\} = J_t u_0 + \int_0^t J_{t-s} E\{g(s)\} ds - \int_0^t \int_0^s R J_{t-s} E\{g(\tau)\} d\tau ds.$$  \hspace{1cm} (15)

Statistical separability occurs on the assumption that is that the input process and system parameters are statistically independent.

Similarly, the correlation function is given by

$$E\{u(t_1)u(t_2)\} = E\{[J_{t_1} u_0 + \int_0^{t_1} J_{t_1-s} g(s) ds - \int_0^{t_1} \int_0^s R J_{t_1-s} g(\tau) d\tau ds]\ \cdot [J_{t_2} u_0 + \int_0^{t_2} J_{t_2-p} g(\rho) d\rho - \int_0^{t_2} \int_0^p R J_{t_2-p} g(\gamma) d\gamma d\beta] \}.$$ \hspace{1cm} (16)

Assuming $g$ is a zero-mean temporal stochastic process and solving,

$$E\{u(t_1)u(t_2)\} = J_{t_1+t_2} u_0^2 + \int_0^{t_1} \int_0^{t_2} \int_0^{t_1} \int_0^{t_2} E\{g(s)g(\rho)\} ds d\rho \cdot d\tau d\xi - 2 \int_0^{t_1} \int_0^{t_2} \int_0^{t_1} \int_0^{t_2} E\{R J_{t_1-s} \gamma g(\tau)\} E\{g(\rho)g(\gamma)\} d\tau d\xi d\rho d\sigma$$ \hspace{1cm} (17)

In Section 4 we shall present an application example when an exact semigroup solution (Section 2) can be found and a comparison with the approximate semigroup solution will be made.
4 Application to modeling groundwater flow with random transmissivity

4.1 The semigroup method

As an example of the application of the semigroup method to the solution of groundwater flow equations with stochastic parameters, consider the one-dimensional groundwater flow equation with Dupuit assumptions in a horizontal aquifer with a fully penetrating stream as a left boundary, and a topographic divide, or no flow, right boundary. This situation arises in a regional alluvial aquifer deposited over a consolidated impervious formation. The governing equation is given by (Bear 1979)

\[
\frac{\partial h(t, \omega)}{\partial t} - \frac{1}{S} \frac{\partial}{\partial x} \left[ K(x, t, \omega) h(x, t, \omega) \frac{\partial h}{\partial x} \right] = \frac{I(x, t, \omega)}{S},
\]

where

\[ h(0, t, \omega) = C_1(t, \omega), \quad \frac{\partial h(x, t, \omega)}{\partial x}(L, t) = C_2(t, \omega), \quad h(x, 0, \omega) = h_0(x, \omega) \]

\[ h(x, t, \omega) \in H^1(0, L) \]

is the hydraulic head (m) or groundwater depth over the bedrock formation; \( x \) is the horizontal distance (m) from the left boundary; \( t \) is the time coordinate in months (mo); \( S \) is the aquifer storage coefficient or specific yield; \( K(t, \omega) \) is the hydraulic conductivity (m.month\(^{-1}\)), and it is assumed a stochastic process; \( I(x, t, \omega) \) is the input function representing deep percolation to the aquifer, and it is assumed a smooth function of \( x \) and random in time (m.month\(^{-1}\)); the functions \( C_1(t, \omega) \) and \( C_2(t, \omega) \) are stochastic processes accounting for the random variations in stream stage and the right boundary respectively; and \( L \) is the total length of the aquifer (m).

The system of equations (18) represents the general stochastic groundwater flow problem of the aquifer in question, where we have a random forcing term, random boundary conditions and a random parameter. At this point the hydrologist must decide which of the random functions are the dominant factors contributing to the stochasticity of the output function \( h \) and which of them could be adequately represented as deterministic functions. This should be done after a careful study of the information available on the stream and groundwater levels, the distribution of precipitation and recharge in space and time and the variation of the measured parameters with time, and keeping in mind that much more information will be required if a function is represented as stochastic than as deterministic because of the required statistical properties. This should not be a reason for neglecting the stochasticity of an input function since such a policy would lead to considerable errors in the output function. Ideally, we could repeatedly solve the problem for each stochastic function, considering the rest as deterministic, and thus test the effect of individual random functions on the overall random behaviour of the output function. Quite often, however, we may detect small order randomness in certain components and relatively large ones in others, so that a particular component may be chosen as the dominant stochastic function.

In the present article we are particularly interested in the solution of the equation with random parameters, but it is quite possible for the method to cover the general case of random initial conditions, random boundary conditions, random forcing term and random parameters as it is described in Section 2. Random boundary conditions ultimately transform into random forcing terms and are described in the previous work (see Serrano, Unny and Lennox, 1985a, b, c). For the purposes of the present article, let us assume that we are interested in modeling the monthly groundwater level for a particular year in order to determine its statistical properties and that the recharge function in Eq. (18) could be approximated as a deterministic step function for which the recharge is constant and uniformly distributed.
across the aquifer at each step. Let us also assume that the left boundary condition is a deterministic constant \( C \) and that the right boundary coincides with the topographic divide and could be assumed as a no flow condition. The product \( Kh \) in Eq. (18) is the aquifer transmissivity \( T \), which is represented as a stochastic process (see Section 1) of the form \( T(t, \omega) = \bar{T}(t) + T'(t, \omega) \), where \( \bar{T}(t) \) is the deterministic seasonal component and \( T'(t, \omega) \) is Gaussian white noise in time (see Section 2) with

\[
E\{T'(t)\} = 0, \quad E\{T'(t_1)T'(t_2)\} = q\delta(t_1 - t_2), \quad t_1 \leq t_2.
\]  

(19)

This last assumption is not necessary, but it is useful for illustration. The correlation structure of \( T' \) should be deduced from field measurements.

Let us now transform our functional space of elements \( h \in \mathcal{H}^1(0, L) \) into the space of elements \( u \in \mathcal{H}^1_0(0, L) \). This is carried out by defining \( u(x, t, \omega) = h(x, t, \omega) - V(x, t, \omega) \), where \( V(x, t, \omega) \) is a “smooth” stochastic function such that \( E\{V(x, \omega)\} \) satisfies the steady state function for a particular time and makes \( u \) vanish at the boundaries. Following the above observations, our transformed system of equations (18) for the case of random transmissivity becomes

\[
\frac{\partial u(x, t, \omega)}{\partial t} + \frac{T(t)}{S} \frac{\partial^2 u(x, t, \omega)}{\partial x^2} = 0, \quad (20)
\]

\[
u(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) = 0, \quad u(x, 0) = h_0(x) - V(x, 0) = u_0(x).
\]

Equation (20) can now be treated as a random evolution equation of the form given by Eq. (3). The function \( V \) is given by

\[
V(x, t, \omega) = \frac{I(t)}{\bar{T}(t, \omega) + T'(t, \omega)}(Lx - \frac{x^2}{2}) + C.
\]  

(21)

Since the recharge function can be approximated as a step function, we may also approximate the deterministic component \( \bar{T}(t) \) as a step function. At each step \( t_i \), \( \bar{T}(t_i) \) will be a constant and \( T'(t_i, \omega) \) will be a random variable. We can solve the equation stepwise and for each \( t_{i-1} < t < t_{i+1} \) Eq. (20) can be treated as a random evolution equation for a time independent operator \( A \), whose solution is given by Eq. (6) as

\[
u(x, t, \omega) = J_{t, 0}u_0, \quad t_{i-1} < t < t_{i+1}
\]  

(22)

where \( u_0 \) is the initial condition at each step, that is the \( u \) at the end of the previous step, and \( J_{t, 0} \) is the random semigroup associated with the partial differential operator in Eq. (20). This semigroup can be easily found by using well-known methods (Curtain and Falb 1971; Curtain and Pritchard 1978) to be

\[
J_{t, 0}u = \sum_{n=1}^{\infty} b_n(u)\Phi_n(x)M_n(t)M_{nr}(t, \omega)
\]  

(23)

where

\[
b_n(u) = \frac{2}{L} \int_{0}^{L} (u)\Phi_n(x)dx,
\]

\[
\Phi_n(x) = \sin(\lambda_n x),
\]

(24)  

(25)
\[ M_n(t) = \exp\left(-\frac{\lambda_n^2 T(t) t}{S}\right). \]  

(26)

\[ M_{nr}(t, \omega) = \exp\left(-\frac{\lambda_n^2}{S} \int_0^t T'(t, \omega) ds\right) = \exp\left(-\frac{\lambda_n^2 \beta(t)}{S}\right). \]  

(27)

In the above \( \beta(t) \) is the Brownian motion or Wiener process (see Jazwinski 1970), and the eigenvalues are

\[ \lambda_n = \left(\frac{2n - 1}{2L}\right) \pi, \quad n = 1, 2, 3, \ldots. \]  

(28)

Note that the semigroup in this case is composed of an exponential function on the deterministic component of the transmissivity and an exponential function on the random component of the transmissivity. Equation (22) represents the stochastic transient function, whose second moment will vanish at \( t = \infty \). Equation (22) can be used to generate sample functions of \( u \), and therefore of the output process \( h \) by using Eq. (21), after generating sample functions of the process \( T' \). Sample functions of \( h \) may be used to study the effect of random variations in transmissivity on the behaviour of the groundwater level \( h \).

The mean value of \( h \) may be obtained by taking expectations on Eqs. (21) and (22):

\[ E\{h(x, t)\} = E\{V(x, t)\} + \sum_{n=1}^\infty b_n(u_0) \Phi_n(x) M_n(t) E\{M_{nr}(t)\} \]  

(29)

where

\[ E\{V(x, t)\} = \frac{I(t)}{T(t)} \left(Lx - \frac{x^2}{2}\right) + C. \]  

(30)

Expanding the random exponential term \( M_{nr}(t) \) as a Taylor series, truncating at the second-order term we find

\[ E\{M_{nr}(t)\} = 1 + \frac{\lambda_n^4 qt}{2S^2}. \]  

(31)

The second moment of \( u \) in Eq. (22) is given by Eq. (9) and Eq. (21). We are particularly interested in an expression for the variance of the groundwater level. This is obtained in Eq. (9) by making \( t_1 = t_2 = t \) and expanding:

\[ E\{h^2(x, t)\} - E^2\{h(x, t)\} = \sigma_h^2 = E\{J_{2t}u_0^2\} - E\{(h(x, t) - V(x, t))^2\} + \sigma_V^2 \]  

(32)

where

\[ \sigma_V^2 = \left(\frac{T(t)^2}{T(t)^2 + q}\right) [E\{V(x, t)\} - C]^2. \]  

(33)

To solve for the first term in the right hand side of Eq. (32), we follow a similar procedure to the one used to derive the mean Eq. (29); that is we replace Eq. (23) in Eq. (32), take expectations, expand the random exponential as a Taylor series, truncate at the second-order term and solve. For the second term in Eq. (32) we simply use Eq. (29). Thus

\[ \sigma_h^2 = \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{n} \Phi_n(x) \Phi_m(x) M_n(t) M_m(t) \left[1 + \frac{qt}{S^2} \left(\frac{\lambda_n^4}{2} + \frac{\lambda_n^2 \lambda_m^2}{2} + \frac{\lambda_m^4}{2}\right)\right] \]
In order to illustrate the above development, a sample problem was solved for an alluvial aquifer, for which a constant hydraulic conductivity could be assumed (see observations in Section 1). Furthermore, \( L \) was assumed equal to 100 m, \( C \) equal to 10 m, \( S \) equal to 0.14, \( T(t) \) equal to 180 m\(^2\) month\(^{-1}\), which corresponds to an average silty soil, and the transmissivity noise parameter \( q \) equal to 81 m\(^2\) month\(^{-1}\). These values of course should be deduced from field measurements and from long-term study of the fluctuations in the groundwater levels. Simple piezometer tests for \( K \) at regular intervals in time and groundwater level observations with time should give enough information to derive the seasonal values and statistical properties of the stochastic transmissivity. In this sample problem it was assumed that the transmissivity could be expressed as the sum of a constant value and a white Gaussian noise of parameter \( q \). This was done for convenience, but seasonal variations of transmissivity should be included in field applications. One year monthly simulations were done beginning in early spring with assumed high values of deep percolation in early spring and ending with zero values of deep percolation in late winter. These values should reflect real recharge as predicted by a water balance like model.

Mean values, sample values and values of variance \( h \), were computed at several points across the aquifer by solving month after month the corresponding equations (29), (22) and (34) respectively. Figure 1 shows the initial condition, mean groundwater levels and a sample function after one month of intense recharge, and Table 1 shows the standard deviation of \( h \) across the aquifer. Note that the standard deviation increases with the distance to the fixed deterministic boundary. Figure 2 and Table 2 show the corresponding values during a dry period six months later. Figure 4 shows a sample transmissivity for the simulated year, and Fig. 3 shows the mean and a sample function of \( h \) at \( x = 50 \) m. We can see that at this aquifer the time random variations in transmissivity may explain the random fluctuations in groundwater level and that the size of fluctuation in heads will depend on the parameter \( q \) of the perturbing stochastic process in \( T \).

4.2 Approximate calculation of the semigroup

In the present example the semigroup method is quite an efficient and simple solution to our problem. However, in many circumstances the derivation of a semigroup may be a complex matter because of the way in which the randomness appear in the partial differential operator.

We shall illustrate the approximation technique exposed in Section 3 with the same example above in order to compare the approximate method against the "exact" one in the previous section.

Let us write Eq. (20) in the form given by Eq. (11):

\[
\frac{\partial u}{\partial t} = \frac{1}{S} \frac{\partial^2 u}{\partial x^2} - \frac{T(t)}{S} \frac{\partial^2 u}{\partial x^2} + \frac{T'(t, \omega)}{S} \frac{\partial^2 u}{\partial x^2},
\]

\( u(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) = 0, \quad u(x, 0) = u_0 \)

where the deterministic components are in the left hand side and the random operator acts as a forcing term. Proceeding in a similar fashion to Section 4.1., we approximate the monthly mean transmissivity \( T(t) \) as a monthly step function so that the partial differential operator is time independent at each time step (each
Table 1. Hydraulic heads at month 1 (m)

<table>
<thead>
<tr>
<th>$x$</th>
<th>Semigroup</th>
<th>Approximate</th>
<th>Semigroup</th>
<th>Approximate</th>
<th>Semigroup</th>
<th>Approximate</th>
</tr>
</thead>
<tbody>
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<td>10.604</td>
<td>10.604</td>
<td>10.671</td>
<td>10.675</td>
<td>0.039</td>
<td>0.040</td>
</tr>
<tr>
<td>20.0</td>
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<td>11.130</td>
<td>11.257</td>
<td>11.264</td>
<td>0.075</td>
<td>0.075</td>
</tr>
<tr>
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<td>11.582</td>
<td>11.762</td>
<td>11.771</td>
<td>0.106</td>
<td>0.106</td>
</tr>
<tr>
<td>40.0</td>
<td>11.965</td>
<td>11.965</td>
<td>12.191</td>
<td>12.203</td>
<td>0.133</td>
<td>0.133</td>
</tr>
<tr>
<td>50.0</td>
<td>12.283</td>
<td>12.283</td>
<td>12.548</td>
<td>12.561</td>
<td>0.155</td>
<td>0.156</td>
</tr>
<tr>
<td>60.0</td>
<td>12.540</td>
<td>12.540</td>
<td>12.836</td>
<td>12.849</td>
<td>0.174</td>
<td>0.175</td>
</tr>
<tr>
<td>70.0</td>
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<td>12.737</td>
<td>13.057</td>
<td>13.070</td>
<td>0.189</td>
<td>0.189</td>
</tr>
<tr>
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<td>12.876</td>
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<td>13.226</td>
<td>0.199</td>
<td>0.200</td>
</tr>
<tr>
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<td>12.959</td>
<td>13.308</td>
<td>13.319</td>
<td>0.205</td>
<td>0.206</td>
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<tr>
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<td>12.987</td>
<td>13.339</td>
<td>13.350</td>
<td>0.207</td>
<td>0.208</td>
</tr>
</tbody>
</table>

Table 2. Hydraulic heads at month 6 (m)

<table>
<thead>
<tr>
<th>$x$</th>
<th>Semigroup</th>
<th>Approximate</th>
<th>Semigroup</th>
<th>Approximate</th>
<th>Semigroup</th>
<th>Approximate</th>
</tr>
</thead>
<tbody>
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<td>10.294</td>
<td>10.312</td>
<td>10.294</td>
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<td>0.005</td>
</tr>
<tr>
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<td>10.573</td>
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<td>0.010</td>
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<tr>
<td>30.0</td>
<td>10.830</td>
<td>10.830</td>
<td>10.884</td>
<td>10.832</td>
<td>0.011</td>
<td>0.015</td>
</tr>
<tr>
<td>40.0</td>
<td>11.064</td>
<td>11.064</td>
<td>11.134</td>
<td>11.066</td>
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<td>0.018</td>
</tr>
<tr>
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<td>11.269</td>
<td>11.354</td>
<td>11.271</td>
<td>0.016</td>
<td>0.021</td>
</tr>
<tr>
<td>60.0</td>
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<td>11.442</td>
<td>11.540</td>
<td>11.445</td>
<td>0.017</td>
<td>0.023</td>
</tr>
<tr>
<td>70.0</td>
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<td>11.580</td>
<td>11.688</td>
<td>11.583</td>
<td>0.019</td>
<td>0.025</td>
</tr>
<tr>
<td>80.0</td>
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<td>11.681</td>
<td>11.797</td>
<td>11.684</td>
<td>0.019</td>
<td>0.026</td>
</tr>
<tr>
<td>90.0</td>
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<td>11.742</td>
<td>11.862</td>
<td>11.745</td>
<td>0.020</td>
<td>0.027</td>
</tr>
<tr>
<td>100.0</td>
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<td>11.762</td>
<td>11.884</td>
<td>11.766</td>
<td>0.020</td>
<td>0.027</td>
</tr>
</tbody>
</table>

month) and we can invoke a semigroup solution for Eq. (35):

$$u = J_t u_0 + \int_{t_{i-1}}^{t} J_{t-s} R u(s) ds, \quad t_{i-1} < t < t_{i+1}$$

(36)

where $R$ is the random partial differential operator $T'(t, \omega) \frac{\partial^2 u}{\partial x^2}$, and where $J_t$ is the semigroup given by the deterministic operator

$$J_t u = \sum_{n=1}^{\infty} b_n(u) \Phi_n(x) M_n(t)$$

(37)

and all of the other terms as before.

Now expand $u$ in the right hand side as a series $\sum u_i$:

$$u = J_t u_0 + \int_{0}^{t} J_{t-s} R (u_1(s) + u_2(s) + u_3(s) + ...) \, ds.$$

(38)

Identifying $u_1$ as the preceding term or the deterministic solution $J_t u_0$, we can determine each $u_i$ in terms of the preceding term $u_{i-1}$. Equation (38) will become an infinite series in which the last term contains $u$ (see Eq. (15)):
Figure 1. Mean and sample free-surface profiles after one month of intense recharge

Figure 2. Mean and sample free-surface profiles during a dry period six months later

Figure 3. Mean and sample functions at x=50m

\[ u = J_t u_0 + \int_{s_0}^{s} RJ_{s-x} R d\xi d\eta + \int_{s_0}^{s} RJ_{s-x} R d\xi d\eta + \cdots \]  

(39)

Because the calculation of each subsequent term in the series is significantly more complex than the previous one, and significantly smaller in size, we shall limit our calculations to the first two terms in the right hand side of Eq. (39).

Sample functions can be computed from Eq. (39) by generating sample functions of \( T' \) (i.e., approximating it as a random walk) and approximating the time integral:
where \( t \) in this case will be equal to 1 because of the stepwise solution,

\[
\begin{aligned}
\sum_{k=0}^{t-1} \exp\left(-\frac{\lambda^2 T(t)(t-k)}{S}\right) \exp\left(-\frac{\lambda^2 T(t)k}{S}\right) T''(\omega, k)
\end{aligned}
\]

\[
h(x, t, \omega) = V(x, \omega) + J_t u_0 - \frac{1}{S} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda^2_t \sin(\lambda_n x)
\]

\[
\sum_{k=0}^{t-1} \exp\left(-\frac{\lambda^2 T(t)(t-k)}{S}\right) \exp\left(-\frac{\lambda^2 T(t)k}{S}\right) T''(\omega, k)
\]

\[
c_{nm} = \frac{2}{L} \left[ \frac{\sin(\lambda_n - \lambda_m) L}{2(\lambda_n - \lambda_m)} - \frac{\sin(\lambda_n + \lambda_m) L}{2(\lambda_n + \lambda_m)} - \frac{1}{2(\lambda_n - \lambda_m)} + \frac{1}{2(\lambda_n + \lambda_m)} \right], \quad n \neq m
\]

\[
c_{nm} = 1 - \frac{\sin(2\lambda_n L)}{2L\lambda_n}, \quad n = m
\]

The expected value of \( h \) from Eq. (39) is given by Eq. (15) and in this case it coincides with the deterministic solution:

\[
E\{h(x, t)\} = E\{V(x, t)\} + J_t u_0.
\]

Finally the variance of \( h \) from Eq. (39) is
In order to illustrate the procedure, the same sample problem solved in Section 4.1 was solved by this approximate calculation technique. The same conceptual aquifer, the same data and the same conditions were reproduced and calculation of the mean monthly groundwater level, monthly sample functions and monthly standard deviations were done by applying the above equations. Tables 1 and 2 show the corresponding results for one wet and one dry month respectively. It is interesting to note that the statistical measures, as computed by both methods are almost identical, while the sample function values are very close to one another. These values were not plotted in Figs. 1 and 2 because the differences between the two methods are not appreciable at this scale. This is an important result since the approximation method will allow us to accurately solve much more complex equations by calculating only the first term in the series. One inconvenience of the approximation method is the amount of computational time required. The approximation method required about three times as much time as required by the exact semigroup method to solve the same problem on a micro-computer.

5 Conclusions

Two methods for the solution of PDE with random in time coefficients were studied and their application to regional groundwater flow problems with random transmissivity were illustrated. The results indicate that a valuable alternative stochastic approach to analyze and solve groundwater equations under general transient conditions follows from a conception of the physical parameters of a dynamic system as randomly fluctuating stochastic processes. It was found that random in time fluctuations in the transmissivity of a phreatic aquifer can explain the transient stochastic behaviour of the output function, that is the groundwater level. Time stochasticity in the parameters constitutes the general case in the analysis of transient groundwater flow problems. One feature of the analysis under time stochasticity, besides generality, includes a much simpler evaluation of the statistical properties of the parameter stochastic processes, i.e., the use of data regularly collected in groundwater studies (simple piezometer tests, groundwater level observations, etc.). This is in contrast with the task of evaluating the spatial statistical properties of hydraulic conductivity as required by time and space stochastic theory.

The semigroup approach for solving a PDE with random coefficients offers a direct procedure giving statistical separability from the physics of the problem rather than limitations on the randomness size or closure approximations assumptions. In this respect the semigroup approach has many advantages over perturbation methods, hierarchy techniques and Montecarlo simulations.

The second method investigated, that is the approximation of the semigroup solution, produces almost identical results to the “exact” semigroup method in the statistical measures of the solution and very similar sample functions. It also produces statistical separability from the physics of the problem. The main feature of this method is that more difficult equations, where randomness enters in a complex
way or there are more than one random parameter, could be accurately solved. There is also a promising perspective for the solution of random PDE in two or three-dimensional domains. One disadvantage of the method is the considerably higher expense in computer time.

Finally, we must add that if a semigroup operator could be derived for a particular partial differential operator in question, the semigroup solution offers an excellent procedure to solve stochastic PDE subject to random initial conditions and/or random boundary conditions and/or random forcing terms and/or random coefficients, and under general transient conditions.

6 References


Sagar, B. 1979: Solution of the linearized Boussinesq equation with stochastic boundaries and recharge. Water Resour. Res. 15, 618-624


Accepted October 14, 1986.