Chapter 4
Receiver Design
Chapter 4
Receiver Design

- Probability of Bit Error
- Pages 124-149
• Probability of Bit Error

The low pass filtered and sampled PAM signal results in an expression for the probability of bit error $P_b$ (S&M p. 124-127). $A$ is the amplitude at the sampling point and $\gamma$ is the attenuation of the channel ($0 \leq \gamma \leq 1$)

$$P\{ \text{ith bit in error} \} = P(b_i = 0) \ P\{ \ n_o[ (i-1)T_b + T_b/2 ] < -\gamma A \}$$
$$+ P(b_i = 1) \ P\{ \ n_o[ (i-1)T_b + T_b/2 ] \geq \gamma A \}$$
• Review of Probability and Stochastic Processes (S&M p. 127-132)

Probability distribution function
\[ F_X(a) = P\{ X = a \} \]

Probability density function
\[ f_X(x) = \frac{d F_X(x)}{dx} \]

Mean (or expected value)
\[ \mu_X = \int x f_X(x) \, dx \]

Variance
\[ \sigma^2_X = \int (x - \mu_X)^2 f_X(x) \, dx \]
\[ E \{ (X - \mu_X)^2 \} \]
• Review of Probability and Stochastic Processes (S&M p. 127-132)

Joint probability distribution function

\[ F_{X,Y}(a, b) = P\{ X = a \text{ and } Y = b \} \]

Joint probability density function

\[ f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \, \partial y} \]

Conditional probabilities

\[ P \{ X > a \text{ and event } Z \} = P \{ \text{event } Z \} \frac{P\{ X > a \mid \text{event } Z \}}{P\{ \text{event } Z \}} \]
Chapter 4

Receiver Design

• *Examining Thermal Noise*

• Pages 132-136
• **Johnson–Nyquist noise** or **thermal noise** is the electronic noise generated by the thermal agitation of the charge carriers (usually the electrons) inside an electrical conductor at equilibrium.

This thermal noise was first measured by John B. Johnson at Bell Labs in 1928. He described his findings to Harry Nyquist, also at Bell Labs, who was able to explain the results.

Harry Nyquist
1889-1976
• Thermal (or Gaussian) noise is approximately white, meaning that the power spectral density is equal throughout the frequency spectrum. Additionally, the amplitude of the signal has very nearly a Gaussian probability density function with mean $\mu_n = 0$.

S&M Figure 4-3

$\mu_n = 0 \quad \sigma_n = 1$
• Since thermal noise has a Gaussian probability density function the probability that a noise voltage $n(t)$ at time $t_o$ will be \textit{less than or equal to a threshold} $-\gamma A$ is (S&M Eq. 4.27):

$$P\{ n(t_o) \leq -\gamma A \} = F_X(-\gamma A) = \int_{-\infty}^{-\gamma A} \frac{1}{\sqrt{2\pi} \sigma_n} \exp\left[ \frac{(x-\mu_n)^2}{2\sigma_n^2} \right] dx$$

and the probability that a noise voltage $n(t)$ at time $t_o$ will be \textit{greater than a threshold} $\gamma A$ is (S&M Eq. 4.28):

$$P\{ n(t_o) > \gamma A \} = 1 - F_X(\gamma A) = \int_{\gamma A}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_n} \exp\left[ \frac{(x-\mu_n)^2}{2\sigma_n^2} \right] dx$$
• The probabilistic properties of thermal noise do not change with time (*stationarity*). Thermal noise is an *insidious property* of communication systems that limits the speed of reliable data transmission and the detection of weak signals.
A MATLAB and Simulink simulation verifies the spectral characteristics of thermal noise and the performance of low-pass filtered $f_{\text{cutoff}} = 11.25$ kHz thermal noise.
• Thermal noise PSD = $N_o$, $|f| \rightarrow \infty$

MS Figure 1.12

• Thermal noise LPF PSD = $N_o$, $|f| < 11.25$ kHz

$N_o$ (single-sided spectrum)

$N_o$

$N_o$ (single-sided spectrum)

$N_o$

11.25 kHz
• Thermal noise PSD = $N_0$, $|f| \rightarrow \infty$

![Thermal noise PSD diagram](image1)

• Thermal noise autocorrelation

![Thermal noise autocorrelation diagram](image2)
• Thermal noise LPF PSD = $N_0$, $|f| < 11.25$ kHz

- LPF thermal noise autocorrelation
  - MS Figure 1.14

- LPF Noise Auto Correlator
  - correlated
• **Simulink** Histogram block from the *Statistics, Signal Processing Blockset* is analogous to the *pdf*. The Vector Scope block from the *Sinks, Signal Processing Blockset*, displays the histogram. **Simulink** blocks to calculate the mean $\mu$ and variance $\sigma^2$ are in *Statistics*.
• Thermal noise histogram display

\[
\mu = 10 \\
\sigma^2 = 1
\]

Gaussian pdf

• LPF \( f_{\text{cutoff}} = 1 \) kHz thermal noise histogram display

\[
\mu = 10.42 \\
\sigma_0^2 = 0.64
\]

pdf remains Gaussian after LPF
• LPF thermal noise has an average normalized power:

\[ \sigma_o^2 = N_o f_{cutoff} \]  

(S&M p. 135)

The probability that the ith-bit is received in error is:

\[
P(b_i = 0) P\{ n(t_o) < -\gamma A \} + P(b_i = 1) P\{ n(t_o) \geq \gamma A \} = \]

\[
P(b_i = 0) \int_{-\infty}^{-\gamma A} \frac{1}{\sqrt{2\pi} \sigma_o} \exp\left[ \frac{(x - \mu_o)^2}{2\sigma_o^2} \right] dx + \]

\[
P(b_i = 1) \int_{\gamma A}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_o} \exp\left[ \frac{(x - \mu_o)^2}{2\sigma_o^2} \right] dx \]
• *If and only if* the binary thresholds are symmetrical ($-\gamma_A$ and $\gamma_A$), the $P(b_i = 0) + P(b_i = 1) = 1$ and the Gaussian normal pdfs are *symmetrical* the probability that the $i$th-bit is received in error becomes (S&M p. 136):

$$P(b_i = 0) P\{n(t_o) < -\gamma_A\} + P(b_i = 1) P\{n(t_o) \geq \gamma_A\} =$$

$$\int_{-\gamma_A}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_o} \exp\left[\frac{(x - \mu_o)^2}{2\sigma_o^2}\right] dx$$
Chapter 4

Receiver Design

- Gaussian Probability Density Function, Probability of Bit Error
- Pages 137-149
• Gaussian (normal) probability density function (pdf)

$$\frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right)$$
• Gaussian (normal) probability distribution function

\[
\frac{1}{2} \left( 1 + \text{erf} \frac{x - \mu}{\sigma \sqrt{2}} \right)
\]
• Gaussian (Normal) Probability Distribution

Abraham de Moivre was a French mathematician famous for de Moivre's formula, which links complex numbers and trigonometry, and for his work on the normal distribution and probability theory in 1734. He wrote a book on probability theory entitled *The Doctrine of Chances* which was said to be highly prized by gamblers.

Gauss rigorously justified and extended the work in 1809.
• Gaussian (Normal) Probability Distribution

Johann Carl Friedrich Gauss was a German mathematician and scientist who contributed significantly to many fields, including number theory, geometry, electrostatics, astronomy and optics.
• Gaussian pdfs

μ = 0, σ = 1
S&M Figure 4-6a

μ = 1.6, σ = 1
S&M Figure 4-6b
• Gaussian pdfs

\[ f_X(x) = \frac{1}{\sqrt{2\pi}(2)} e^{-\frac{x^2}{2(2)^2}} \]

\( \mu = 0, \sigma = 2 \)
S&M Figure 4-6c

\( \mu = 1, \sigma = 2 \)
S&M Figure 4-6d
• The **probability of bit error** is the **area** under the Gaussian pdf from the threshold $a$ to $\infty$ which could be tabulated. However, the probability of bit error is determined by three independent variables ($a$, $\mu$ and $\sigma$) and this would be an unwieldy table.

S&M Figure 4-7
• Rather, construct a single table with $\mu = 0$ and $\sigma = 1$ for the probability of bit error as the area under the Gaussian pdf as a function of the threshold $a$ only which is known as the complementary error or Q-function.

S&M Figure 4-7
• The Q-function for $\mu = 0$ and $\sigma = 1$ as a function of the threshold $a$ is listed in Table 4-1 (S&M p. 141) and Appendix B (MS p. 185-186)
• Reading the $Q$-function table correctly

• What is $Q(1.82)$?
• What is $Q(2.63)$?
• What is $Q(3.18)$?
• Reading the **Q-function** table correctly

- Q(1.82) = 0.0344
- Q(2.63) = 0.0043
- Q(3.18) = 0.0007
• What if $\mu \neq 0$? It can be shown that the Q-function table remains valid if the threshold variable in the table is changed from $a$ to $a - \mu$. Note that the areas under the Gaussian pdfs are the same.

S&M Figure 4-8a
Figure 4-8b
• What if $\mu \neq 0$ and $\sigma \neq 1$? It can also be shown that the Q-function table remains valid if the threshold variable in the table is first changed from $a$ to $a - \mu$ and …

S&M Figure 4-9a
Figure 4-9b
• ... then the threshold variable in the Q-function table is changed from $a - \mu$ to $(a - \mu) / \sigma$. Note that argument on the $x$-axis is compressed by $\sigma$ and the $y$-axis is expanded by $\sigma$.  

S&M Figure 4-9b
Figure 4-9c
For simple baseband PAM the probability of bit error $P_b$ is expressed by the Q-function:

$$P_b = Q\left( \frac{\text{noise margin of sampled value}}{\sqrt{\text{average normalized noise power at the input to the single point sampler}}} \right)$$

Sampling at $T_b/2$
Chapter 4

Receiver Design

• Optimal Receiver: The Matched Filter or Correlation Receiver

• Pages 149-161
• The simple baseband PAM receiver structure is:

\[ r(t) \rightarrow \text{Low Pass Filter} \begin{array}{c} f_c = \text{bandwidth of} \\ \text{transmitted signal} \end{array} \rightarrow z(t) \rightarrow \text{Sample in center of} \\ \text{each bit period} \rightarrow \text{Threshold} \\ \text{comparison} \]

But is this the best that there is? What about sampling an odd number of times (like 3) during each bit time \( T_b \)?
• Although sampling can be increased to a very large, odd number of samples during $T_b$, there is an optimal way:

Since LPF in PAM improved performance, assume that the optimal processing is a linear filter $H(f)$ (S&M p. 150)
• After development (S&M p. 150-153) the optimal processing is a linear filter $H_0(f)$

The optimal linear filter $H_0(f)$ has an impulse response $h_0(t)$ and is known as a *matched filter* since the processing is matched to input signal $s(t)$:

$$H_0(f) = k S^*(f) \exp(-j 2\pi f i T_b)$$

$$h_0(t) = F^{-1}\{H_0(f)\} = k F^{-1}\{S^*(f) \exp(-j 2\pi f i T_b)\}$$

$$h_0(t) = k s(i T_b - t) \quad \text{impulse response matched to } s(t)$$
• The impulse response of the optimum filter $h_o(t)$ is a scaled (by $k$), time delayed (by $iT_b$) and time reversed (function of $iT_b - t$):

$$h_o(t) = k s(iT_b - t)$$
• When optimum processing is used the argument inside the Q-function is maximized (S&M p. 153-154) and the probability of bit error $P_b$ is:

$$P_b = Q \left( \max \frac{|r(iT_b) \ast h(iT_b)|}{\sigma_p} \right)$$

$$P_b = Q \left( \sqrt{\max \frac{|r(iT_b) \ast h(iT_b)|^2}{\sigma_p^2}} \right)$$

$$P_b = Q \left( \sqrt{\frac{2E_b}{N_o}} \right) \quad \text{S&M Eq. 4.58}$$

where $E_b$ is the *energy per bit* of the received signal.
The optimum filter $H_o(f)$ is equivalent to the correlation receiver (S&M p. 155-156).

**Optimum Filter**

$$h_o(t) = k s(iT_b - t)$$

**Correlation Receiver**

$$z(iT_b) = \int_{-(i-1)T_b}^{iT_b} r(t) s_1(t) dt$$
• Since the optimum filter $H_0(f)$ and the correlation receiver are equivalent, with $s_1(t) = s(t)$ for a matched filter and $r(t) = \gamma s(t)$ where $\gamma$ is the communication channel attenuation, the energy-per-bit $E_b$ is (S&M p. 156, Eq 4.62):

$$E_b = \int_{(i-1)T_b}^{iT_b} \gamma s(t) \gamma s(t) \, dt = \gamma^2 \int_{(i-1)T_b}^{iT_b} s^2(t) \, dt$$

\[ z(iT_b) = \int_{(i-1)T_b}^{iT_b} r(t) s_1(t) \, dt \]
The expected or mean value $a_i(iT_b)$ is the output of the correlation receiver when $r(t) = \gamma s_i(t)$ and $n(t) = 0$ where $\gamma$ is the communication channel attenuation.

$$a_i(iT_b) = \int_{(i-1)T_b}^{iT_b} \gamma s_i(t) s_1(t) \, dt$$

S&M Eq. 4.67
• $P_b = Q(\sqrt{2E_b / N_o})$ and the ratio $E_b / N_o$ can be expressed in dB: $10 \log_{10}(E_b / N_o)$. The resulting plot of $P_b$ verses $E_b / N_o$ in dB is a characteristic of binary symmetric PAM with AWGN.

S&M Figure 4-14
• Symmetric binary PAM implies that the two transmitted signals for binary 1 and binary 0 $s(t)$ and the resulting outputs $a(iT_b)$ from the correlation receiver are equal in magnitude but opposite in sign:

$$s_{bi=1}(t) = -s_{bi=0}(t)$$

S&M Figure 4-15
• The probability of bit error for *equally-likely* binary symmetric PAM is the sum of the *error regions* shown.

\[ P(b_i=0) = P(b_i=1) = 0.5 \]

S&M Figure 4-16
• The binary PAM signals are symmetrical and the threshold is 0 (equidistant from the means or expected values ± a(iT_b)). The error regions are equal in area.

S&M Figure 4-16
• The probability of bit error does not minimize if the correlation receiver threshold is \textit{misadjusted} ($\tau \neq 0$). With a misadjusted threshold the apriori probabilities are now important since the area of the error regions are no longer equal.

S&M Figure 4-17
The correlation receiver is also known as the \textit{integrate-and-dump} which describes the process.

\[ z(iT_b) = \int_{(i-1)T_b}^{iT_b} r(t) s_1(t) \, dt \]

\text{Threshold comparison}

\text{matched filter or correlation receiver}

MS Figure 2.29
Chapter 2

Baseband Modulation and Demodulation

• Optimum Binary Baseband Receiver: The Correlation Receiver

• Pages 40-41
• The correlation receiver can be simulated in Simulink:

\[ z(iT_b) = \int_{(i-1)T_b}^{iT_b} r(t) s_1(t) \, dt \]
• The Simulink Integrate and Dump block is in the Comm Filters, Communications Blockset.
The parameters of the Simulink Integrate and Dump block are the integration period and offset in simulation samples.

\[ f_{\text{simulation}} = 50 \text{ kHz} \]
\[ T_{\text{simulation}} = \frac{1}{f_{\text{simulation}}} = 0.02 \text{ msec} \]

\[ T_b = 1 \text{ msec} \]

Integration period = \[ T_b / T_{\text{simulation}} = 1 / 0.02 = 50 \text{ samples} \]

Offset = 0 samples
• The complete binary symmetrical rectangular PAM digital communication system with BER analysis and optimum receiver.

MS Figure 2.30
- Observed BER as a function of SNR for binary rectangular PAM in a LPF simple receiver (LPF) and the optimum correlation receiver (CR) with normalized signal power = 25 W (MS Table 2.3 p. 27 and Table 2.6 p. 43).

<table>
<thead>
<tr>
<th>SNR dB</th>
<th>AWGN $\sigma^2$ V^2</th>
<th>BER (LPF)</th>
<th>BER (CR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3.98</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.96</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-3.52</td>
<td>50</td>
<td>$1 \times 10^{-4}$</td>
<td>0</td>
</tr>
<tr>
<td>-6.02</td>
<td>100</td>
<td>$6.3 \times 10^{-3}$</td>
<td>$1 \times 10^{-4}$</td>
</tr>
<tr>
<td>-9.03</td>
<td>200</td>
<td>$3.63 \times 10^{-2}$</td>
<td>$6.4 \times 10^{-3}$</td>
</tr>
<tr>
<td>-13.01</td>
<td>500</td>
<td>$1.185 \times 10^{-1}$</td>
<td>$6.02 \times 10^{-2}$</td>
</tr>
<tr>
<td>-16.02</td>
<td>1000</td>
<td></td>
<td>$1.345 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
The energy per bit $E_b = 2.5 \times 10^{-2} \text{ V}^2\text{-sec}$ (S&M Eq. 4.62, p. 156) for rectangular $\pm 5 \text{ V}$ PAM with the channel attenuation $\gamma = 1$:

$$E_b = \int_{(i-1)T_b}^{iT_b} \gamma s(t) \gamma s(t) \, dt = \gamma^2 \int_{(i-1)T_b}^{iT_b} s^2(t) \, dt$$

The observed bit error rate (BER) can be compared to the theoretical probability of bit error $P_b$ (S&M Eq. 4.58, p. 154) to validate the basic simulation.

$$P_b = Q \left( \frac{2E_b}{N_0} \right)$$
The ratio of the energy per bit $E_b$ to the noise power spectral density $N_o (E_b / N_o)$ rather than the SNR is the conventional metric used for BER performance.

$$E_b = \int_{(i-1)T_b}^{iT_b} y s(t) y s(t) \, dt = \gamma^2 \int_{(i-1)T_b}^{iT_b} s^2(t) \, dt$$

$$P_b = Q \left( \sqrt{\frac{2E_b}{N_o}} \right)$$

S&M Eq. 4.58, S&M Eq. 4.62
• The BER and $P_b$ comparison (MS Table 2.7, p. 44):

Table 2.7 Observed BER and Theoretical $P_b$ as a Function of $E_b / N_o$ in a Binary Symmetrical Rectangular PAM Digital Communication System with Optimum Receiver

<table>
<thead>
<tr>
<th>$E_b / N_o$ dB</th>
<th>BER</th>
<th>$P_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>$4.05 \times 10^{-6}$</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>$2.06 \times 10^{-4}$</td>
</tr>
<tr>
<td>6</td>
<td>$2.5 \times 10^{-3}$</td>
<td>$2.43 \times 10^{-3}$</td>
</tr>
<tr>
<td>4</td>
<td>$1.35 \times 10^{-2}$</td>
<td>$1.25 \times 10^{-2}$</td>
</tr>
<tr>
<td>2</td>
<td>$3.95 \times 10^{-2}$</td>
<td>$3.75 \times 10^{-2}$</td>
</tr>
<tr>
<td>0</td>
<td>$8.09 \times 10^{-2}$</td>
<td>$7.93 \times 10^{-2}$</td>
</tr>
</tbody>
</table>
Chapter 4

Receiver Design

• Correlation Receiver for Asymmetric PAM, Optimum Thresholds, Synchronization

• Pages 162-173
Asymmetric PAM signals do not have equal means or expected values of the output of the correlation receiver:

\[ |a_2(iT_b)| \neq |a_1(iT_b)| \]
• The optimum threshold $\tau_{opt}$ is again equidistant between the means or expected values of the output of the correlation receiver:

$$T_{opt} = \frac{a_2(iT_b) + a_1(iT_b)}{2}$$

S&M Eq. 4.71

Here $|a_2(iT_b)| + \tau_{opt} = a_1(iT_b) - \tau_{opt}$ and the threshold is equidistant from the means or expected values.
• If the apriori probabilities are not equal then the optimum threshold $\tau_{opt}$ is not equidistant from the means or expected value of output of the correlation receiver. An asymmetric binary PAM signal is shown:

$$a_2(iT_b) \neq a_1(iT_b)$$

S&M Figure 4-23
• The optimum threshold $\tau_{opt}$ where the apriori probabilities are $(1 - M)$ and $M$ (which sums to 1) is:

$$T_{opt} = \frac{2 \sigma_o^2 \ln \left( \frac{M}{(1-M)} \right) + a_2(iT_b)^2 - a_1(iT_b)^2}{2 \left[ a_2(iT_b) - a_1(iT_b) \right]}$$

S&M Eq. 4.85

if $M = 0.5$ then:

$$T_{opt} = \frac{a_2(iT_b) + a_1(iT_b)}{2}$$

S&M Eq. 4.71
Asymmetric PAM with an optimum threshold $\tau_{opt}$ has a probability of bit error $P_b$:

$$P_b = Q\left(\frac{a_1(iT_b) - a_2(iT_b)}{2 \sigma_o}\right)$$

$$P_b = Q\left(\sqrt{\frac{a_1(iT_b) - a_2(iT_b)}{4 \sigma_o^2}}\right)^2$$

$$P_b = Q\left(\sqrt{\frac{E_d}{2 N_o}}\right)$$

where $E_d = \int_{(i-1)T_b}^{iT_b} \left\{ \gamma \left(s_1(t) - s_2(t)\right) \right\}^2 dt$

S&M Figure 4-19

$$P_b = \frac{1}{2} P[z(iT_b) < \tau_{opt} | b_i = 1] + \frac{1}{2} P[z(iT_b) \geq \tau_{opt} | b_i = 0]$$

$$\sigma_o^2 = \frac{N_o}{2}$$

S&M Eq. 4.78 and Eq. 4.79
• The optimum correlation receiver for asymmetric binary PAM uses the difference signal $s_1(t) - s_2(t)$ as the reference:

$$S(t) = r(t) [s_1(t) - s_2(t)]$$

$$z(iT_b) = \int r(t) [s_1(t) - s_2(t)] dt$$
• The optimum correlation receiver can be reconfigured as an alternate but universal structure which can be used for both asymmetric or symmetric binary PAM signals:

![Diagram of the optimum correlation receiver](image-url)
If the *apriori probabilities* are not equal then the optimum threshold $\tau_{opt}$ is not equidistant from the means or expected value of output of the correlation receiver. An asymmetric binary PAM signal is shown:

$$a_2(iT_b) \neq a_1(iT_b)$$

S&M Figure 4-23
• The optimum threshold $\tau_{opt}$ where the a priori probabilities are $(1 - M)$ and $M$ (which sums to 1) is:

$$T_{opt} = \frac{2 \sigma^2 \ln \left( \frac{M}{1 - M} \right) + a_2(iT_b)^2 - a_1(iT_b)^2}{2 \left[ a_2(iT_b) - a_1(iT_b) \right]}$$

S&M Eq. 4.85

if $M = 0.5$ then:  

$$T_{opt} = \frac{a_2(iT_b) + a_1(iT_b)}{2}$$

S&M Eq. 4.71
The probability of bit error $P_b$ then is:

\[ P_b = M Q \left( \frac{a_1(iT_b) - T_{opt}}{\sigma_0} \right) + (1-M) Q \left( \frac{T_{opt} - a_2(iT_b)}{\sigma_0} \right) \]

\[ P_b = M Q \left( \sqrt{\frac{(a_1(iT_b) - T_{opt})^2}{2N_0}} \right) + (1-M) Q \left( \sqrt{\frac{(T_{opt} - a_2(iT_b))^2}{2N_0}} \right) \]
• If the apriori probabilities are equal ($M = 0.5$):

$$T_{\text{opt}} = \frac{a_2(iT_b) + a_1(iT_b)}{2}$$

and $P_b$ becomes:

$$P_b = 0.5 Q \left( \sqrt{ \frac{(a_1(iT_b) - T_{\text{opt}})^2}{2 N_o} } \right) + 0.5 Q \left( \sqrt{ \frac{(T_{\text{opt}} - a_2(iT_b))^2}{2 N_o} } \right)$$

$$P_b = Q \left( \sqrt{ \frac{(a_1(iT_b) - a_2(iT_b))^2}{2 N_o} } \right) = Q \left( \sqrt{ \frac{E_d}{2 N_o} } \right)$$

S&M p. 168
The BER and $P_b$ comparison (MS Table 2.8, p. 49):

Table 2.8 Observed BER and Theoretical $P_b$ as a Function of $E_d/N_o$ in an Asymmetrical Binary Rectangular PAM Digital Communication System with Optimum Receiver.

<table>
<thead>
<tr>
<th>$E_d/N_o$ dB</th>
<th>BER</th>
<th>$P_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>$2.5 \times 10^{-3}$</td>
<td>$2.53 \times 10^{-3}$</td>
</tr>
<tr>
<td>10</td>
<td>$1.28 \times 10^{-2}$</td>
<td>$1.25 \times 10^{-2}$</td>
</tr>
<tr>
<td>8</td>
<td>$3.59 \times 10^{-2}$</td>
<td>$3.75 \times 10^{-2}$</td>
</tr>
<tr>
<td>6</td>
<td>$8.05 \times 10^{-2}$</td>
<td>$7.93 \times 10^{-2}$</td>
</tr>
<tr>
<td>4</td>
<td>$1.334 \times 10^{-1}$</td>
<td>$1.318 \times 10^{-1}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.856 \times 10^{-1}$</td>
<td>$1.872 \times 10^{-1}$</td>
</tr>
<tr>
<td>0</td>
<td>$2.362 \times 10^{-1}$</td>
<td>$2.394 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
• Since for binary rectangular PAM $E_d = 4 E_b$ (S&M p. 168) the BER performance for asymmetric PAM is comparable to symmetric PAM if a $E_d / N_o$ is reduced by 6 dB $(10 \log (4) = 6)$:

<table>
<thead>
<tr>
<th>$E_d / N_o$ dB</th>
<th>BER</th>
<th>$P_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asymmetric PAM</td>
<td>10</td>
<td>$1.28 \times 10^{-2}$</td>
</tr>
<tr>
<td>(MS Table 2.8,</td>
<td>8</td>
<td>$3.59 \times 10^{-2}$</td>
</tr>
<tr>
<td>p. 49)</td>
<td>6</td>
<td>$8.05 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$E_b / N_o$ dB</th>
<th>BER</th>
<th>$P_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric PAM</td>
<td>4</td>
<td>$1.35 \times 10^{-2}$</td>
</tr>
<tr>
<td>(MS Table 2.7,</td>
<td>2</td>
<td>$3.95 \times 10^{-2}$</td>
</tr>
<tr>
<td>p. 44)</td>
<td>0</td>
<td>$8.09 \times 10^{-2}$</td>
</tr>
</tbody>
</table>
Chapter 2

Baseband Modulation and Demodulation

• The Correlation Receiver for Baseband Asymmetrical Signals

• Pages 44-47
• The optimum threshold $\tau_{opt}$ requires the additive Gaussian noise variance $\sigma^2$ as processed by the correlation receiver or $\sigma_o^2$:

$$
T_{opt} = \frac{2 \sigma_o^2 \ln \left( \frac{M}{1-M} \right) + a_2(iT_b)^2 - a_1(iT_b)^2}{2 \left[ a_2(iT_b) - a_1(iT_b) \right]}
$$
The optimum threshold $T_{opt}$ also requires the apriori probabilities $P_1 = M$ and $P_0 = M - 1$:

$$T_{opt} = \frac{2 \sigma_o^2 \ln \left( \frac{M}{1-M} \right) + a_2(iT_b)^2 - a_1(iT_b)^2}{2 \left[ a_2(iT_b) - a_1(iT_b) \right]}$$

Here $P_1 = 0.4998 = M$ and $P_0 = 1 - M = 0.5004$
• The Mean and Variance blocks are in the **Statistics, Signal Processing** Blockset.
• The parameter of the Mean block can be set for running mean or an average over the simulation period. The result can be shown by a Display block from the Simulink Blockset.
• The parameter of the Variance block can be set for running variance or an average over the simulation period. The result can be shown by a Display block from the Simulink Blockset.
The Simulink simulation for binary asymmetrical PAM with BER analysis has a threshold adjustment based on the estimate of $\tau_{opt}$ from measurements of $\sigma_o$ and $M$. The threshold function is given by:

$$T_{opt} = \frac{2\sigma_o^2 \ln \left( \frac{M}{1-M} \right) + a_2(iT_b)^2 - a_1(iT_b)^2}{2 \left[ a_2(iT_b) - a_1(iT_b) \right]}$$

MS Figure 2.33
• The *Simulink* simulation for the alternative and universal structure for the correlation receiver:
• Synchronization (not considered here) provides *timing recovery* or the exact beginning and end of a bit time $T_b$: 

![S&M Figure 4-21](image)

**start of $T_b$?**
• Synchronization of both **symbol time** and carrier frequency and phase is one of the advanced topics in EE4542 Telecommunications Engineering. The effect is called **jitter**.
Chapter 4

Receiver Design

• *Multi-level PAM (M-ary PAM)*

• Pages 200-206
• Multi-level (M-ary) PAM is another means to minimize the bandwidth required for a data transmission rate $r_b$ b/sec. Rather than transmitting a binary signal in a bit time $T_b$, send a multi-level (usually a power-of-2) signal during the same period called the symbol time $T_s$.

A multi-state comparator determines the received symbol which is then decoded to the received bits.

S&M Figure 4-48

\[
M = 4
\]
The three optimum thresholds if the apriori probabilities are equally likely \((P_i = 0.25)\) are:

\[
T_{\text{opt}1} = \frac{a_1(iT_S) + a_2(iT_S)}{2}
\]

\[
T_{\text{opt}2} = \frac{a_2(iT_S) + a_3(iT_S)}{2}
\]

\[
T_{\text{opt}3} = \frac{a_3(iT_S) + a_4(iT_S)}{2}
\]
• The probability of symbol error $P_s$, where $M = 2^n$ is the number of levels, can be shown to be:

$$P_s = \frac{2(M - 1)}{M} Q\left(\sqrt{\frac{E_{d,symbol}}{2N_0}}\right)$$

where:

$$E_{d,symbol} = \int_{(i-1)T_s}^{iT_s} \left\{ y\left[ s_j(t) - s_k(t) \right] \right\}^2 dt$$
• There are $2(M-1)$ error regions due to only adjacent regions being misinterpreted with $M = 2^n$ equally probable symbols. The probability of occurrence $P_j$ for a misinterpreted symbol is also equally likely and is:

$$P_j = \frac{1}{2(M-1)}$$

SVU Eq. 2.41
The amplitudes of the M-ary (M = 4) rectangular PAM signal is ±3A and ±A. The energy difference for a symbol \( E_{d,symbol} = 4\gamma^2 A^2 T_S \) (S&M Eq. 4.140):

\[
E_{d,symbol} = \int_{(i-1)T_s}^{iT_s} \{ \gamma \left[ s_j(t) - s_k(t) \right] \}^2 dt
\]

The average received energy per symbol \( E_{avg,symbol} = 5\gamma^2 A^2 T_S \) (S&M Eq. 4.141)
Then \( E_{d,\text{symbol}} = 0.4 \ E_{avg,\text{symbol}} \) and substitute \( (M = 4) \):

\[
P_s = \frac{3}{2} Q \left( \sqrt{\frac{E_{d,\text{symbol}}}{2 N_o}} \right) \quad \text{S&M Eq. 4.139b}
\]

\[
P_s = \frac{3}{2} Q \left( \sqrt{\frac{0.4 \ E_{avg,\text{symbol}}}{N_o}} \right) \quad \text{S&M Eq. 4.142a}
\]
The average energy per bit $E_b = 0.5 E_{avg,symbol}$ since $M = 4$ and there are two bits per symbol:

$$P_s = \frac{3}{2} Q\left(\sqrt{\frac{0.4 E_{avg,symbol}}{N_o}}\right)$$  \hspace{1cm} \text{S&M Eq. 4.142a}$$

$$P_s = \frac{3}{2} Q\left(\sqrt{\frac{0.8 E_b}{N_o}}\right)$$  \hspace{1cm} \text{S&M Eq. 4.142b}$$

S&M Figure 4-50 modified
• The symbols transmitted can have 1 bit in error $\frac{4}{6}$ of the time and 2 bits in error $\frac{2}{6}$ of the time:

<table>
<thead>
<tr>
<th>Transmitted Di-Bit</th>
<th>Received Di-Bit</th>
<th>Bits In Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>01</td>
<td>1</td>
</tr>
<tr>
<td>01</td>
<td>00</td>
<td>1</td>
</tr>
<tr>
<td>01</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>01</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
P_{b,4\text{-level}} = \frac{4}{6} P(1\text{ of 2 bits in error}) + \frac{2}{6} P(2\text{ of 2 bits in error})
\]

\[
P_{b,4\text{-level}} = \frac{4}{6} \frac{1}{2} P_s + \frac{2}{6} \frac{2}{3} P_s = \frac{2}{3} P_s \quad \text{S&M Eq. 4.143}
\]
The probability of bit error $P_{b,4\text{-symbol}} = 2 \frac{P_s}{3}$ but can that be improved? Change the assignment of symbols to di-bits as a Gray code and there is only 1 bit in error for each of the six error regions and $P_{b,4\text{-symbol}} = \frac{P_s}{2}$.

<table>
<thead>
<tr>
<th>Transmitted Di-Bit</th>
<th>Received Di-Bit</th>
<th>Bits In Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>00</td>
<td>1</td>
</tr>
<tr>
<td>00</td>
<td>01</td>
<td>1</td>
</tr>
<tr>
<td>00</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>00</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

$P_{b,4\text{-level}} = \frac{6}{6} P(1 \text{ of 2 bits in error})$

$P_{b,4\text{-level}} = \frac{6}{6} \frac{1}{2} P_s = \frac{1}{2} P_s$
The simple change to a Gray coded symbol improves the probability of bit error $P_b$:

$$P_b = \frac{2 \ 3 \ 3 \ 2}{3 \ 2} \quad Q \left( \frac{0.8 \ E_b}{N_o} \right) = \frac{3}{4} \quad Q \left( \frac{0.8 \ E_b}{N_o} \right)$$

Straight binary coded

Gray coded

S&M Figure 4-50
Frank Gray was a researcher at Bell Labs who made numerous innovations in television and is remembered for the **Gray code**. The Gray code appeared in his 1953 patent and is a binary system often used in electronics. Gray also conducted pioneering research on the development of television. He proposed an early form of the **flying spot** scanning system for TV cameras in 1927, and helped develop a two-way mechanically scanned TV system in 1930.
Chapter 2

Baseband Modulation and Demodulation

- Multilevel (M-ary) Pulse Amplitude Modulation

- Pages 49-55
The Simulink simulation of a 4-level straight binary (not Gray coded) rectangular PAM system with BER analysis:
Although Simulink provides a Bit to Integer block and an Integer to Bit block from the Utility Blocks, Communication Blockset, these blocks use a vector of bits rather than from and to a serial bitstream.
• A bit to symbol converter Simulink subsystem can be configured from Downsampler blocks and a Delay block from the Signal Operations, Signal Processing Blockset and the Gain and Sum blocks from the Simulink, Math Operations to produce the 4-level PAM signal (0, 1, 2 or 3) from the input di-bit (00, 01, 10, or 11) with the most significant bit (MSB) first.

MS Figure 2.36
• The *Simulink* bit to symbol converter can be easily reconfigured to produce an 8-level PAM signal (0, 1, 2, … 7) from the input *tri-bit* (000, 001, … 111) with the most significant bit (MSB) first.
A symbol to bit converter Simulink subsystem can be configured from Constant blocks, Pulse Generator blocks and a Multiport Switch block from the Signal Routing, Simulink Blockset to produce the di-bit (MSB first) from the 4-level PAM signal.

MS Figure 2.40
A symbol to bit converter *Simulink* subsystem can be easily reconfigured to produce the *tri-bit* (MSB first) from an 8-level PAM signal.

---

**Figure 2.41**

![Diagram](image)
• Binary data to 4-level PAM converter

MS Figure 2.36
• Transmit binary data, $r_b = 1 \text{ kb/sec}$

• Transmit 4-level PAM signal, $r_S = 500 \text{ symbols/sec}$
• M-ary PAM transmits \( n \) bits per symbol \( (M = 2^n) \) but has the same rectangular pulse shape as binary PAM. The normalized power spectral density for M-ary PAM \( \text{PSD}_M \) has the same \( \text{sinc} \) shape as that for binary PAM \( \text{PSD}_B \) but uses the symbol time \( T_S \) rather than the bit time \( T_b \):

\[
\text{PSD}_M(f) = A_{\text{avg}}^2 T_S \text{sinc} \left( \pi T_S f \right)
\]

\[
\text{PSD}_B(f) = A^2 T_b \text{sinc} \left( \pi T_b f \right)
\]
• Observed BER and theoretical $P_b$ as a function of $E_b / N_o$ in a 4-level rectangular straight-binary coded PAM digital communication system with optimum receiver (MS Table 2.11 p. 58).

<table>
<thead>
<tr>
<th>$E_b / N_o$ dB</th>
<th>BER</th>
<th>$P_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>$2.7 \times 10^{-3}$</td>
<td>$2.4 \times 10^{-3}$</td>
</tr>
<tr>
<td>8</td>
<td>$1.28 \times 10^{-2}$</td>
<td>$1.25 \times 10^{-2}$</td>
</tr>
<tr>
<td>6</td>
<td>$3.42 \times 10^{-2}$</td>
<td>$3.75 \times 10^{-2}$</td>
</tr>
<tr>
<td>4</td>
<td>$7.52 \times 10^{-2}$</td>
<td>$7.78 \times 10^{-2}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.291 \times 10^{-1}$</td>
<td>$1.320 \times 10^{-1}$</td>
</tr>
<tr>
<td>0</td>
<td>$1.842 \times 10^{-1}$</td>
<td>$1.867 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
The M-ary PAM PSD uses the average amplitude $A_{\text{avg}}$, where $P_j$ is the a priori probability of occurrence of the amplitude $A_j$.

$$A_{\text{avg}}^2 = \sum_{j=1}^{M} A_j^2 P_j$$
• The M-ary (M = 4) PAM PSD can be determined by a Simulink model

![Simulink model](Fig239.mdl)

**MS Figure 2.39**
The Simulink simulation of a 4-level straight binary (not Gray coded) rectangular PAM system measures the BER performance.
• The rectangular M-ary pulse width is the entire symbol time $T_S$ and is optimum in the bandwidth sense. The bandwidth is similar to that for binary PAM but with $T_b = T_S$ (cf. S&M Table 3-1 p. 86).

Table 2-9 Bandwidth of a Binary Rectangular M-ary PAM Signal as a Percentage of the Total Power (MS p. 53)

<table>
<thead>
<tr>
<th>Bandwidth (Hz)</th>
<th>Percentage of Total Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/T_S$</td>
<td>90%</td>
</tr>
<tr>
<td>$1.5/T_S$</td>
<td>93%</td>
</tr>
<tr>
<td>$2/T_S$</td>
<td>95%</td>
</tr>
<tr>
<td>$3/T_S$</td>
<td>96.5%</td>
</tr>
<tr>
<td>$4/T_S$</td>
<td>97.5%</td>
</tr>
<tr>
<td>$5/T_S$</td>
<td>98%</td>
</tr>
</tbody>
</table>
• $\text{PSD}_M$  $M = 4$, $r_b = 1 \text{ kb/sec}$, first null bandwidth $= 500 \text{ Hz}$

• $\text{PSD}_B$  $r_b = 1 \text{ kb/sec}$, first null bandwidth $= 1 \text{ kHz}$
Chapter 2

Baseband Modulation and Demodulation

- Performance of M-ary Pulse Amplitude Modulation
- Gray Encoded Data
- Pages 57-60
• The Simulink simulation of a 4-level straight binary Gray encoded rectangular PAM system with BER analysis:

MS Figure 2.42
• The M = 4 level symbol (0, 1, 2 and 3) is Gray encoded by a **Lookup Table Block** from the **Simulink Blockset**.
The M = 4 level symbol (0, 1, 2 and 3) is Gray encoded by a Lookup Table Block from the Simulink Blockset.
The M = 4 level symbol (0, 1, 2, and 3) is Gray encoded by a Lookup Table Block by mapping [0, 1, 2, 3] to [0, 1, 3, 2].

Gray coding:
- 00 → 00
- 01 → 01
- 10 → 11
- 11 → 10
• The M = 4 level Gray encoded symbol is decoded as di-bits by a Lookup Table Block from the Simulink Blockset.
The \( M = 4 \) level symbol (0, 1, 2, and 3) is Gray decoded by a Lookup Table Block by mapping [0, 1, 2, 3] to [0, 1, 3, 2].

- Gray coding:
  - 00 → 00
  - 01 → 01
  - 10 → 11
  - 11 → 10
The PSD of a Gray coded rectangular PAM signal is the same as a straight-binary rectangular PAM signal. The coding does not change the PSD. The PSD is only affected by the pulse shape and data rate.

\[ \text{PSD}_M \ M = 4, \ r_b = 1 \text{ kb/sec}, \ \text{first null bandwidth} = 500 \text{ Hz} \]
The Simulink simulation of a 4-level straight binary Gray encoded rectangular PAM system measures the BER performance. MS Figure 2.42
• Observed BER and theoretical $P_b$ as a function of $E_b / N_o$ in a 4-level rectangular Gray coded PAM digital communication system with optimum receiver (MS Table 2.12 p. 60).

<table>
<thead>
<tr>
<th>$E_b / N_o$ dB</th>
<th>BER</th>
<th>$P_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>$1.7 \times 10^{-3}$</td>
<td>$1.8 \times 10^{-3}$</td>
</tr>
<tr>
<td>8</td>
<td>$8.6 \times 10^{-3}$</td>
<td>$9.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>$2.92 \times 10^{-2}$</td>
<td>$2.81 \times 10^{-2}$</td>
</tr>
<tr>
<td>4</td>
<td>$5.93 \times 10^{-2}$</td>
<td>$5.84 \times 10^{-2}$</td>
</tr>
<tr>
<td>2</td>
<td>$9.56 \times 10^{-2}$</td>
<td>$9.90 \times 10^{-2}$</td>
</tr>
<tr>
<td>0</td>
<td>$1.441 \times 10^{-1}$</td>
<td>$1.404 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
• Observed BER as a function of $E_b / N_o$ for binary rectangular PAM for straight-binary coded (SBC) and Gray coded (GC) data (MS Table 2.11 p. 58 and MS Table 2.12 p. 60).

<table>
<thead>
<tr>
<th>$E_b / N_o$ dB</th>
<th>BER (SBC)</th>
<th>BER (GC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>$2.7 \times 10^{-3}$</td>
<td>$1.7 \times 10^{-3}$</td>
</tr>
<tr>
<td>8</td>
<td>$1.28 \times 10^{-2}$</td>
<td>$8.6 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>$3.42 \times 10^{-2}$</td>
<td>$2.92 \times 10^{-2}$</td>
</tr>
<tr>
<td>4</td>
<td>$7.52 \times 10^{-2}$</td>
<td>$5.93 \times 10^{-2}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.291 \times 10^{-1}$</td>
<td>$9.56 \times 10^{-2}$</td>
</tr>
<tr>
<td>0</td>
<td>$1.842 \times 10^{-1}$</td>
<td>$1.441 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Although a slight improvement, Gray coding improves BER performance without an increase in $E_b$. 
End of Chapter 4
Receiver Design