Chapter 2

Frequency Domain Analysis
Chapter 2

Frequency Domain Analysis

• Why Study Frequency Domain Analysis?

• Pages 6-13
Why frequency domain analysis?

- Allows simple algebra rather than time-domain differential equations to be used.
- Transfer functions can be applied to transmitter, communication channel and receiver.
- Channel bandwidth, noise and power are easier to evaluate.

MS Figure 4.2 and Figure 4.3
Why frequency domain analysis?

• A complex signal consisting of the sum of three sinusoids is difficult to discern in the temporal domain but easy to identify in the spectral domain.

MS Figure 4.2 and Figure 4.3

500, 1500 and 2500 Hz
• Example 2.1 Input sum of three sinusoids, temporal display

Butterworth LPF
9 pole, $f_{cutoff} = 1 \text{ kHz}$

EX21.mdl

MS Fig 4.8 modified
• Example 2.1  Input sum of three sinusoids

• Output after Butterworth LPF
• Example 2.1  Input sum of three sinusoids, temporal display

Since this is a MATLAB and Simulink time-based simulation, an analog low pass filter is used here

Butterworth LPF
9 pole, $f_{cutoff} = 1 \text{ kHz}$

EX21.mdl

MS Fig 4.8 modified
• Example 2.1  Input sum of three sinusoids, spectral display

500 Hz
Sine Wave

1500 Hz
Sine Wave

2500 Hz
Sine Wave

Butterworth LPF
9 pole, $f_{cutoff} = 1$ kHz

EX21spectral.mdl

MS Fig 4.11 modified
• Input power spectral density of the sum of three sinusoids

Fig. 11mod/Sinusoid Spectrum

34 dB

• Output power spectral density after Butterworth LPF

Fig. 11mod/LPF Sinusoid Spectrum

Attenuation (decibel dB)

-36 dB
• **Butterworth Filters**

Stephen Butterworth was a British physicist who invented the Butterworth filter, a class of circuits that are used to filter electrical signals. He worked for several years at the National Physical Laboratory (UK), where he did theoretical and experimental work for the determination of standards of electrical inductance and analyzed the electromagnetic field around submarine cables.
Example 2.2

10 MHz sinusoid with additive white Gaussian noise (AWGN)

Communications channel

EX22.mdl
• Example 2.2  *Simulink* design window

Configure Simulation  Simulate

EX22.mdl

T = 0.01 sec

EX22.mdl
Example 2.2  Simulink design window

Configure Simulation

\[ f_{\text{simulation}} = \frac{1}{T_{\text{simulation}}} = 10^{-8} = 10 \text{ nsec} \]

\[ T_{\text{simulation}} = 10^{8} = 100 \text{ MHz} \]

\[ T = 0.01 \text{ sec} \]
Example 2.2  10 MHz sinusoid with AWGN

Power spectral density of 10 MHz sinusoid with AWGN

Here’s the signal!
Chapter 2

Frequency Domain Analysis

• *The Fourier Series*

• Pages 13-38
• Fourier Series

Jean Baptiste Joseph Fourier was a French mathematician and physicist who is best known for initiating the investigation of Fourier Series and its application to problems of heat flow. The Fourier transform is also named in his honor.

\[ s(t) = X_0 + \sum_{n=1}^{\infty} X_n \cos(2\pi n f_0 t + \varphi_n) \]
• Fourier series coefficients:
  - trigonometric: $a_n$, $b_n$
  - polar: $X_n$
  - complex: $c_n$

\[ X_0 = a_0 \quad X_n = \sqrt{a_n^2 + b_n^2} \]
\[ |c_n| = X_n / 2 \quad X_n = |2c_n| \]

\[ X_0 = c_0 \]

MATLAB and Simulink simulation can provide the magnitude of the complex Fourier series coefficients for any periodic waveform.
Example 2.3 modified

Temporal display of a complex pulse as the addition of two periodic pulses

EX23.mdl
Example 2.3 modified • Simulink design window

Configure Simulation    Simulate

T = 4 sec

EX23.mdl
Example 2.3 modified

Simulink design window

Configure Simulation

\[ T_{\text{simulation}} = \frac{1}{f_{\text{simulation}}} = 0.976562 \text{ msec} \quad T = 4 \text{ sec} \]
• Example 2.3 modified

Simulink discrete pulse generators

EX23.mdl

Temporal display scopes

First Pulse

Sum

Second Pulse
Example 2.3 modified

\[ f_s = 1024 \text{ Hz} \]

\[ T_{\text{simulation}} = \frac{1}{f_s} = 0.976562 \text{ msec} \]

EX23.mdl
• Example 2.3 modified

\[ f_S = 1024 \text{ Hz} \]

\[ T_{\text{simulation}} = \frac{1}{f_{\text{simulation}}} = 0.976562 \text{ msec} \]
• Example 2.3 modified

Period $T_o = 4$ sec
Fundamental frequency $f_o = 0.25$ Hz

Sample based simulation:
Period = 4096 samples
must be $2^N$ for FFT

Pulse width = 1024 samples
Sample time = 0.978562 msec
$4/4096 = 9.78562 \times 10^{-4}$
Example 2.3 modified

The default display or *autoscaling* for the temporal scope does not have a uniform amplitude.

EX23.mdl
- Example 2.3 modified

**Right-Click** on the vertical axis and use **Axes properties** to change the amplitude scaling.
• Example 2.3 modified  First periodic pulse

T = 1 sec

T = 4 sec

• Second periodic pulse

T = 2 sec
• Example 2.3 modified  Sum of periodic pulses
• Example 2.3 modified

Spectral display of a complex pulse as the addition of two periodic pulses
- Example 2.3 modified

\[ f_S = 1024 \text{ Hz} \]

\[ T = 4 \text{ sec} = 4096 \text{ samples} \]

\[ f_S = f_{\text{simulation}} \text{ for convenience} \]

Input signal is non-framed based so buffer input is required
• Example 2.3 modified

\[ f_s = 1024 \text{ Hz} \]

\[ T = 4 \text{ sec} = 4096 \text{ samples} \]

Amplitude scaling \textit{magnitude-squared}
Example 2.3 modified

Scaled $|FFT|^2$ using Simulink Plot Tools. In Command Window:

```matlab
>> plottools on
```

Default spectral display
Example 2.3 modified

Add data point markers (o) and change the frequency axis to 5 Hz.
Chapter 2
Frequency Domain Analysis

- *Power in the Frequency Domain*
- Pages 38-52
**Example 2.3 modified**  Unscaled | FFT $|^2$

- $f_s / 2 = 512$ Hz

**Scaled | FFT $|^2$

- Spectral Resolution  MS Eq 1.1
  - $\Delta f = f_s / N = 1024/4096 = 0.25$ Hz
  - $5$ Hz
• Example 2.3 modified Scaled | FFT |²

The Fourier series components are discrete. However, the Spectrum Scope does not read out the values. An interactive cursor is available in the Figures window.

Copy and Paste the Spectrum Scope plot as Figure 1
Example 2.3 modified Scaled | FFT |²

The interactive cursor can lock-on to the markers and read out the data values sequentially. The interactive cursor is evoked from the toolbar for a Figure plot.

\[ x = 0 \ (f = 0 \text{ Hz}) \]
\[ y = 2362 = | \text{FFT}(f = 0) |² \]
Example 2.3 modified Scaled | FFT |²

The magnitude squared data values are divided by the number of points in the FFT \( N = 4096 \) to get the value. The magnitude of the DC (0 Hz) value is then:

\[
(2362/4096)^{0.5} = (0.56767)^{0.5} = 0.7594 = c_0 = X_0
\]

The DC value of the complex pulse is calculated as 0.75
• Example 2.3 modified \( Scaled \mid FFT \mid^2 \)

The first five spectral components can be read, computed and then compared to a direct calculation of \( c_n \).

<table>
<thead>
<tr>
<th>( f ) Hz</th>
<th>( Data )</th>
<th>( n )</th>
<th>( Component )</th>
<th>Spectral Resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2362</td>
<td>0</td>
<td>0.7594</td>
<td>MS Eq 1.1</td>
</tr>
<tr>
<td>0.25</td>
<td>1038</td>
<td>1</td>
<td>0.5034</td>
<td>( \Delta f = f_s / N )</td>
</tr>
<tr>
<td>0.5</td>
<td>103.8</td>
<td>2</td>
<td>0.1592</td>
<td>( \Delta f = 1024/4096 )</td>
</tr>
<tr>
<td>0.75</td>
<td>115.3</td>
<td>3</td>
<td>0.1678</td>
<td>( \Delta f = 0.25 ) Hz</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

\( \Delta f \) denotes the spectral resolution.
• Example 2.7

Rectangular pulse train

Diagram:
- Pulse Generator
- Scope
- B-FFT
- Spectrum Scope
Example 2.7

Amplitude = 3
Period $T_o = 10$ msec
Fundamental frequency $f_o = 100$ Hz
Pulse width = 2 msec
Duty cycle = $2/10 = 0.2$

Sample based simulation:
Period = 4096 samples
Pulse width = 819 samples
$819/4096 \approx 0.2$
Sample time = $2.441 \mu\text{sec}$
$10^{-2}/4096 = 2.441 \times 10^{-6}$
• Example 2.7

Period $T_o = 10$ msec

Since the Spectrum Scope requires $2^N = 4096$ samples, the simulation time = 10 msec divided by the sample time = 2.441 $\mu$sec must be $\geq 4096$

$10^{-2}/2.441 \times 10^{-6} = 4096.68$
Example 2.7 One cycle of periodic pulse

Magnitude squared of the Fast Fourier Transform $|\text{FFT}|^2$

$T_{\text{pulse}} = 2 \text{ msec}$

DC value = 0.6

$T_o = 10 \text{ msec}$

$|\text{FFT} (f = 0)|^2 = 1447$

$(1447/4096)^{0.5} = (0.353271)^{0.5} = 0.5944$
Example 2.7 \( |\text{FFT}|^2 \)

First spectral null at \( 1/T_{\text{pulse}} = 1/2 \text{ msec} = 500 \text{ Hz} \)

Successive nulls at \( n/T_{\text{pulse}} = n \times 500 \text{ Hz} \)

Hard to discern when the nulls are when plotted on a linear scale
• Example 2.7 $|\text{FFT}|^2$

First spectral null at $1/T_{\text{pulse}} = 1/2 \text{ msec} = 500 \text{ Hz}

Successive nulls at $n/T_{\text{pulse}} = n \times 500 \text{ Hz}$

Easier to see when the nulls are plotted on a decibel (dB) scale where $-40 \text{ dB} \approx 0$
• de·ci·bel (děsˈə-bəl, -běl')
  n. (Abbr. dB)
  A unit used to express relative difference in power or intensity, usually between two acoustic or electric signals, equal to ten times the common logarithm of the ratio of the two levels.

The bel (B) as a unit of measurement was originally proposed in 1929 by W. H. Martin of Bell Labs. The bel was too large for everyday use, so the decibel (dB), equal to 0.1 B, became more commonly used.
• Example 2.7 | FFT |²

Spectrum Scope axis properties

[Diagram of signal flow from Pulse Generator to Spectrum Scope through FFT, with a screenshot of the Spectrum Scope block parameters window showing settings like Frequency units, Frequency range, Frequency display offset, Sample time of original time series, Y-axis scaling, Minimum Y-limit, Maximum Y-limit, and Y-axis title.]
Example 2.7 | FFT $|X|^2 (\text{Data}/4096)^{0.5}$

<table>
<thead>
<tr>
<th>$f$ Hz</th>
<th>Data</th>
<th>Component $n$</th>
<th>Calculated $c_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1447</td>
<td>0.5944</td>
<td>0.6</td>
</tr>
<tr>
<td>100</td>
<td>1290</td>
<td>0.5612</td>
<td>0.561</td>
</tr>
<tr>
<td>200</td>
<td>844</td>
<td>0.4540</td>
<td>0.454</td>
</tr>
<tr>
<td>300</td>
<td>376</td>
<td>0.3030</td>
<td>0.303</td>
</tr>
<tr>
<td>400</td>
<td>81</td>
<td>0.1406</td>
<td>0.140</td>
</tr>
<tr>
<td>500</td>
<td>≈ 0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$X = a + b$

$|c_n| = X_n / 2$

$X_n = |2c_n|$
Example 2.8

Rectangular pulse trains
with pulse period of 0.5 sec
and pulse widths of
0.0625 sec
0.125 sec
0.250 sec

Sample based simulation:
Period = $2^{16} = 65536$ samples
Pulse widths = 8192, 16384, 32768
Sample time = $7.629 \mu\text{sec}$
$0.5/65536 = 7.629 \times 10^{-6}$

EX28.mdl
Example 2.8

Spectrum Scope has inherent amplitude and frequency axes scaling (but not cursor readout)
First null 16 Hz = $1/\tau$ \hspace{1cm} \tau = 0.0625 \text{ sec}

First null 8 Hz = $1/\tau$ \hspace{1cm} \tau = 0.125 \text{ sec}

First null 4 Hz = $1/\tau$ \hspace{1cm} \tau = 0.250 \text{ sec}

S&M p. 36-37
• Average Normalized \((R_L = 1\Omega)\) Power

Fourier series components from simulation

\begin{align*}
X_0 &= a_0 \\
X_n &= \sqrt{a_n^2 + b_n^2} \\
|c_n| &= X_n / 2 \\
X_n &= |2c_n|
\end{align*}

Periodic signal as a frequency domain representation

\[ s(t) = X_0 + \sum_{n=1}^{\infty} X_n \cos(2\pi f_0 t + \phi_n) \]

Average \((R_L = 1\Omega)\) power in the signal \(P_S\) as a time domain or frequency domain representation

\[ P_S = \frac{1}{T} \int_{t_0}^{t_0+T} s^2(t) \, dt = X_0^2 + \sum_{n=1}^{\infty} \frac{X_n^2}{2} \]

*Parseval’s Theorem*
• Average ($R_L \neq 1\Omega$) Power

The average non-normalized ($R_L \neq 1\Omega$) power in the signal $P_S$ as a time domain or frequency domain representation

$$s(t) = X_0 + \sum_{n=1}^{\infty} X_n \cos(2\pi f_0 t + \phi_n)$$

$$P_S = \frac{1}{TR_L} \int_{t_0}^{t_0+T} s^2(t) \, dt = \frac{1}{R_L} \left[ X_0^2 + \sum_{n=1}^{\infty} \frac{X_n^2}{2} \right]$$

*Parseval’s Theorem*
• Parseval’s Theorem

Marc-Antoine Parseval des Chênes was a French mathematician, most famous for what is now known as Parseval’s Theorem, which presaged the equivalence of the Fourier Transform. A monarchist opposed to the French Revolution, Parseval fled the country after being imprisoned in 1792 by Napoleon for publishing tracts critical of the government.

\[
P_s = \frac{1}{T} \int_{t_0}^{t_0+T} s^2(t) \, dt = X_0^2 + \sum_{n=1}^{\infty} \frac{X_n^2}{2}
\]
• Examples 2.9 and 2.10

Normalized power spectrum of a rectangular pulse and a Butterworth low pass filtered (LPF) rectangular pulse

EX29.mdl
• Examples 2.9 and 2.10

Amplitude = 3
Period $T_o = 10$ msec
Fundamental frequency $f_o = 100$ Hz
Pulse width = 2 msec
Duty cycle = $2/10 = 0.2$
Sample based simulation:
Period = 4096 samples
Pulse width = 819 samples
$819/4096 \approx 0.2$
Sample time = $10^{-2}/4096 = 2.441 \times 10^{-6} = 2.441$ µsec
Sampling rate = $1/2.441 \times 10^{-6} \approx 409.6$ kHz

EX29.mdl
• Examples 2.9 and 2.10

Since this is a MATLAB and Simulink sample based simulation (period = 4096 samples, pulse width = 819 samples) a digital low pass filter design is used here (a time-based simulation requires an analog filter design)
Example 2.9 and 2.10 Digital Low Pass Filter Design

\[ f_{\text{cuttoff}} = 300 \text{ Hz} \]

\[ f_s = 409600 \]

The attenuation at cutoff frequencies is fixed at 3 dB (half the passband power).
• **Power Spectral Density**

The normalized \((R_L = 1 \, \Omega)\) power spectral density (PSD) for periodic signals is *discrete* because of the fundamental frequency \(f_o = 1/T_o = 100 \, \text{Hz}\) here.

However, for *aperiodic* signals the PSD is conceptually *continuous*. Periodic signals contain *no information* and only aperiodic signals are, in fact, communicated.
• Power Spectral Density

Simulated by $|\text{FFT}|^2$ in MATLAB and Simulink to obtain $c_n$ from which we can derive $X_n$

$$|\text{FFT}|^2 \approx \text{PSD}$$

$$X_0 = a_0, \quad X_n = \sqrt{a_n^2 + b_n^2}$$

$$X_0 = c_0, \quad |c_n| = X_n / 2, \quad X_n = |2c_n|$$
- **Power Spectral Density** Simulated by $|\text{FFT}|^2$

The dB scale for PSD is more prevalent for analysis.
• Power Spectral Density     Low Pass Filtered

**Graphs:**
- **Unfiltered:**
  - Frequency: 0 to 2 kHz
  - dB Scale
  - Magnitude-squared dB

- **LPF (Low Pass Filtered):**
  - Frequency: 0 to 2 kHz
  - dB Scale
  - Magnitude-squared dB
  - $f_{cutoff} = 300 \text{ Hz}$
• Bandwidth

The *bandwidth* of a signal is the width of the frequency band in Hertz that contains a sufficient number of the signal’s frequency components to reproduce the signal with an acceptable amount of distortion.

Bandwidth is a nebulous term and communication engineers must always define what if meant by *bandwidth* in the context of use.
• Total Power in the Signal

Parseval’s Theorem allows us to determine the total normalized power in the signal without the infinite sum of Fourier series components by integrating in the temporal domain:

\[ P_S = \frac{1}{T} \int_{t_0}^{t_0+T} s^2(t) \, dt = X_0^2 + \sum_{n=1}^{\infty} \frac{X_n^2}{2} \]

From Example 2.10 the total normalized \((R_L = 1\Omega)\) power in the signal is 1.8 \(V^2\) (not \(W\)) and the percentage of the total power in the signal in a bandwidth of 300 Hz is approximately 88% (S&M p. 45).
Chapter 2

Frequency Domain Analysis

• *The Fourier Transform*

• Pages 52-69
Example 2.12

Spectrum of a simulated single pulse from a very low duty cycle rectangular pulse train

EX212.mdl
Example 2.12  Simulated single pulse

- Pulse width = 1 msec
- Pulse period = 1 sec
- Duty cycle = $10^{-3}/1 = 0.001 = 0.1\%$

\[ |\text{FFT}|^2 \]
Example 2.12  Simulated single pulse

The magnitude of the Fourier Transform of a single pulse is continuous and not discrete since there is no Fourier series representation. In the MATLAB and Simulink simulation here the data points are very dense and virtually display a continuous plot.
• Example 2.12  Simulated single pulse | FFT |²

S&M p. 56-60

\[ S(f) = A_T \text{sinc}(\pi f \tau) \]

\[ A_T = 1(10^{-3}) = 10^{-3} \]

\[ (c_o/65536)^{0.5} = 10^{-3} \]

\[ c_o = 0.065536 \]
Example 2.12  Simulated single pulse | FFT $|^2$

S&N p. 56-60

S(f) = $A\tau \text{sinc}(\pi f \tau)$

Zero-crossing at integral multiples of $1/\tau = 1/10^{-3} = 1000$ Hz
Example 2.12  Simulated single pulse

duty cycle $= 10^{-2}/1 = 0.01 = 1%$

pulse width $= 10$ msec

pulse period $= 1$ sec

$|FFT|^2$
• Example 2.12  Simulated single pulse | FFT |²

S&M p. 56-60

\[ S(f) = A\tau \text{sinc}(\pi f \tau) \]

Zero-crossing at integral multiples of \(1/\tau = 1/10^{-2} = 100\,\text{Hz}\)
• Properties of the Fourier Transform

- **Linearity**
  \[ af_1(t) + bf_2(t) \overset{\mathcal{F}}{\leftrightarrow} aF_1(\omega) + bF_2(\omega) \]

- **Convolution**
  \[ f_1(t) \ast f_2(t) \overset{\mathcal{F}}{\leftrightarrow} F_1(\omega)F_2(\omega) \]

- **Conjugation**
  \[ \overline{f(t)} \overset{\mathcal{F}}{\leftrightarrow} \overline{F(-\omega)} \]

- **Scaling**
  \[ f(at) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{|a|}F\left(\frac{\omega}{a}\right), \quad a \in \mathbb{R}, a \neq 0 \]

- **Time reversal**
  \[ f(-t) \overset{\mathcal{F}}{\leftrightarrow} F(-\omega) \]

- **Time shift**
  \[ f(t - t_0) \overset{\mathcal{F}}{\leftrightarrow} e^{-i\omega t_0}F(\omega) \]
• Properties of the Fourier Transform

Modulation (multiplication by complex exponential)
\[ f(t) \cdot e^{i\omega_0 t} \quad \mathcal{F} \quad F(\omega - \omega_0), \quad \omega_0 \in \mathbb{R}, \]

Multiplication by \( \sin \omega_0 t \)
\[ f(t) \sin \omega_0 t \quad \mathcal{F} \quad \frac{i}{2}[F(\omega + \omega_0) - F(\omega - \omega_0)] \]

Multiplication by \( \cos \omega_0 t \)
\[ f(t) \cos \omega_0 t \quad \mathcal{F} \quad \frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)] \]

Integration
\[ \int_{-\infty}^{t} f(u) \, du \quad \mathcal{F} \quad \frac{1}{i\omega} F(\omega) + \pi F(0) \delta(\omega) \]

Parseval's theorem
\[ \int_{\mathbb{R}} f(t) \cdot \overline{g(t)} \, dt = \int_{\mathbb{R}} F(\omega) \cdot \overline{G(\omega)} \, d\omega \]
Chapter 2

Frequency Domain Analysis

- *Normalized Energy Spectral Density*
- Pages 60-65
• **Normalized Energy**

If \( s(t) \) is a *non-periodic, finite energy signal* (a single pulse) then the average normalized power \( P_S \) is 0:

\[
P_S = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0 + T} s^2(t) \, dt = \lim_{T \to \infty} \frac{\text{finite value}}{T} = 0 \quad V^2
\]

\[
E_S = \int_{-\infty}^{\infty} s^2(t) \, dt \quad V^2 \text{ - sec}
\]

However, the normalized energy \( E_S \) for the same \( s(t) \) is *non-zero by definition* (S&M p. 60-61).
• Parseval’s Energy Theorem

Parseval’s energy theorem follows directly then from the discussion of power in a periodic signal:

$$E_S = \int_{-\infty}^{\infty} s^2(t) \, dt = \int_{-\infty}^{\infty} |S^2(f)| \, df \quad V^2 \cdot \text{sec}$$

• Energy Spectral Density

Analogous to the power spectral density is the energy spectral density (ESD) $\psi(f)$. For a linear, time-invariant (LTI) system with a transfer function $H(f)$, the output ESD which is the energy flow through the system is:

$$\psi_{\text{OUT}}(f) = \psi_{\text{IN}}(f) \cdot |H(f)|^2$$
• **Energy Spectral Density**

The energy spectral density (ESD) $\psi(f)$ is the magnitude squared of the Fourier transform $S(f)$ of a pulse signal $s(t)$:

$$\psi(f) = |S(f)|^2$$

The ESD can be *approximated* by the magnitude squared of the Fast Fourier Transform (FFT) in a *MATLAB* and *Simulink* simulation as described in Chapter 3.

$$\psi(f) \approx |\text{FFT}|^2$$
End of Chapter 2

Frequency Domain Analysis