A GENERAL DECISION THEORETIC FORMULATION OF PROCEDURES CONTROLLING FDR AND FNR FROM A BAYESIAN PERSPECTIVE

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Abstract: In this paper, the problems of controlling false discoveries and false non-discoveries are considered from a decision-theoretic perspective. A general decision theoretic formulation is given to multiple testing allowing descriptions of measures of false discoveries and false non-discoveries in terms of certain loss functions even when randomized decisions are made on the hypotheses. Randomized as well as non-randomized procedures controlling the Bayes false discovery rate (BFDR) and Bayes false non-discovery rate (BFNR) are developed. The proposed procedures are applicable in any situation, unlike the corresponding frequentist procedures, such as the Benjamini-Hochberg procedure and its FNR analog, which control the BFDR or BFNR but do so under independence or a particular form of positive dependence structure of the test statistics. Even in the presence of this type of positively dependent test statistics, as simulations show, the proposed procedures perform much better than the corresponding frequentist procedures. Under the iid setup, the proposed procedures provide better control of the BFDR or BFNR than the ones where the control is achieved through local FDR or local FNR.

Key words and phrases: One-step randomized procedures; Posterior false discovery rate; Posterior false non-discovery rate

1. Introduction. A tremendous growth of research has taken place recently in the area of multiple hypothesis testing because of its increased relevance in analyzing high-dimensional data. Among several possible measures of overall error rates, the false discovery rate (FDR) and false non-discovery rate (FNR) have received the most attention. The concepts of FDR and FNR have been studied and procedures controlling them have been developed from both frequentist and Bayesian perspectives [Benjamini and Hochberg (1995, 2000), Benjamini and Liu (1999), Benjamini and Yekutieli (2001), Efron (2003), Efron and Tibshirani

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The Bayesian theory of false discoveries and false non-discoveries have been developed to a large extent under a simple model, the so-called iid mixture model, in which the test statistics are assumed iid given a set of null hypotheses that are all true or all false, with the null hypotheses being true or false according to iid Bernoulli random variables. This theory has been further developed in this article using a much more general framework starting with a decision theoretic formulation of multiple testing and using a model where the underlying test statistics and the associated parameters are assumed dependent. We consider, in particular, the problems of controlling the Bayes FDR (BFDR) and Bayes FNR (BFNR). The present formulation of multiple testing is slightly more general than Cohen and Sackrowitz (2005a,b) in that it allows us to define different overall measures combining false discoveries or false non-discoveries more generally in terms either randomized or non-randomized decisions on the null hypotheses.

We provide procedures that control the BFDR or BFNR at a designated level. The non-randomized version of our BFDR (or BFNR) procedure rejects (or accepts) every family of null hypotheses with the average posterior probability of the null (or alternative) hypotheses less (or greater) than a threshold, while the corresponding randomized version, a one-step randomized procedure, allows one additional random rejection (for BDFR control) or acceptance (for BFNR control) providing a slightly better control of the BFDR or BFNR. Our non-randomized BFDR procedure is same as in Muller et al. (2004).

It is important to note that under the aforementioned iid mixture model a control of the BFDR at \( \alpha \) can be achieved by rejecting every null hypothesis whose posterior probability is less than \( \alpha \) [see, for example, Efron et al. (2001)]. However, under a quite common stochastic increasing property of the test statistics, it is more conservative than our proposed BFDR procedure in that it allows less rejections of the null hypotheses. Also, while the frequentist FDR-controlling procedures, such as the most commonly used BH procedure [Benjamini and Hochberg (1995)], also control the BFDR; they require certain
specific distributional assumptions of the test statistics, like positive regression dependence, given the parameters [Benjamini and Yekutieli (2001), Sarkar (2002, 2004)]. In other words, our procedure has a more general applicability. Moreover, as noted through simulation, it is more powerful compared to the BH procedure and that controlling the Bayesian FDR due to Efron (2003), especially when there is dependence in the tests. Efron’s Bayesian FDR is different from the present BFDR. It is related to the notion of positive FDR of Storey (2002) and developed based on the aforementioned iid mixture model. In addition, we also show that one can use Bayesian variable selection procedures [George and McCulloch (1993)] to estimate the same optimal rules in a fashion that is completely different from the iid mixture model.

In order to compare different BFDR controlling procedures, we use the BFNR as a Bayesian performance measure. However, in some applications, controlling false negatives may be of primary importance. Our BFNR controlling procedure is an appropriate multiple testing method in such a situation from a Bayesian perspective. Of course, there are frequentist FNR procedures [Sarkar (2004, 2006)] which also control the BFNR; but, again they require specific distributional assumptions. The present BFNR procedure is an alternative to theses procedures having a more general applicability. The BFDR, in these situations, can be used as a performance measuring criterion.

The layout of this paper is as follows. The decision theoretic formulation of multiple testing, with representations of false discovery and false non-discovery rates in terms of loss functions, is presented Section 2. In Section 3, formulas of Bayesian measures related to false discoveries and false non-discoveries are given. A one-step randomized procedure is introduced in Section 4, along with the formulas of BFDR and BFNR of this and some non-randomized stepwise procedures. Procedures controlling BFDR and BFNR are developed in Section 5. Assuming normal distributions of the test statistics conditional on the parameters, simulation studies are conducted in Section 6 to compare our proposed BFDR controlling procedure with the Benjamini-Hochberg procedure and the procedure controlling the Bayesian FDR of Efron et al. (2001) in three different situations representing different assumptions about the dependence structure of the test statistics. We present a brief explanation of a FDR-based Bayesian
variable selection procedure in Section 7 and conclude with some discussion in Section 8.

2. A Decision theoretic formulation of multiple testing. Suppose that we have a set of random variables $\mathbf{X} = (X_1, \ldots, X_n) \sim P_{\theta}$, $\theta = (\theta_1, \ldots, \theta_n) \in \Theta \subseteq \mathbb{R}^n$, which is being used to test

$$H_i : \theta_i \in \Theta_{i0} \text{ against } K_i : \theta_i \in \Theta_{i1},$$

simultaneously for $i = 1, \ldots, n$.

Let $d_i = 0$ or 1 correspond to the decision of accepting or rejecting $H_i$. Then, $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{D}$, where $\mathbb{D}$ is the decision space given $\mathbf{x}$. Given $\mathbf{d}$, we consider choosing the decision vector $\mathbf{d}$ according to the following probabilities:

$$\delta(\mathbf{d} | \mathbf{x}) = \prod_{i=1}^{n} \left\{ \delta_i(\mathbf{x}) \right\}^{d_i} \left\{ 1 - \delta_i(\mathbf{x}) \right\}^{1-d_i}, \mathbf{d} \in \mathbb{D},$$

for some $0 \leq \delta_i(\mathbf{x}) \leq 1$, $i = 1, \ldots, n$. The vector $\delta(\mathbf{X}) = (\delta_1(\mathbf{X}), \ldots, \delta_n(\mathbf{X}))$ is referred to as a multiple decision rule or multiple testing procedure. If $0 < \delta_i(\mathbf{X}) < 1$, for at least one $i$, then $\delta(\mathbf{X})$ is randomized; otherwise, it is non-randomized.

The main objective in a multiple testing problem is to determine $\delta(\mathbf{X})$, the choice of which is typically assessed based on a risk measured by averaging a loss $L(\theta, \delta)$ it incurs in selecting $\mathbf{d}$ over uncertainties. In a frequentist approach, only the uncertainty in $\mathbf{X}$ given $\theta$ is considered, while in a Bayesian approach, one would like to further utilize prior information on $\theta$.

Let $\mathbf{h} = (h_1, \ldots, h_n)$, with $h_i = 0$ or 1 according as $\theta_i \in \Theta_{i0}$ or $\theta_i \in \Theta_{i1}$, represent the unknown configuration of true or false null hypotheses. Given $Q(\mathbf{h}, \mathbf{d})$, a measure of error providing an overall discrepancy between $\mathbf{h}$ and $\mathbf{d}$, the loss function, is given by

$$L(\theta, \delta(\mathbf{X})) = \sum_{\mathbf{d} \in \mathbb{D}} Q(\mathbf{h}, \mathbf{d}) \delta(\mathbf{d} | \mathbf{X}).$$
The frequentist risk is given by
\[ R_\delta(\theta) = E_{X|\theta} L(\theta, \delta(X)). \] (4)

Given a prior distribution of \( \theta \), the posterior risk is
\[ \pi_\delta(X) = E_{\theta|X} L(\theta, \delta(X)), \] (5)
and the Bayes risk is
\[ r_\delta = E_\theta R_\delta(\theta) = E_X \pi_\delta(X). \] (6)

Among several possible choices of \( Q(h, d) \) providing different frequentist concepts of error rate in multiple testing, we will concentrate on the following:

False Discovery Proportion (FDP):
\[ Q_1(h, d) = \frac{\sum_{i=1}^n d_i(1-h_i)}{\{\sum_{i=1}^n d_i\} \lor 1}, \] (7)

False Non-Discovery Proportion (FNP):
\[ Q_2(h, d) = \frac{\sum_{i=1}^n (1-d_i)h_i}{\{\sum_{i=1}^n (1-d_i)\} \lor 1}, \] (8)

and consider determining \( \delta \) controlling the corresponding Bayes risks, the Bayes False Discovery Rate (BFDR) and the Bayes False Non-Discovery Rate (BFNR), respectively. Often in practice, where controlling false positives is of primary importance, finding a \( \delta \) that controls the BFDR would be the main objective, and the BFNR could be used as a performance measuring criterion to compare different BFDR controlling procedures. In some applications, however, one wants to control false negatives, rather than false positives. The roles of the BFDR and BFNR can be switched in these situations.

**Remark 2.1.** It is important to note that for a non-randomized decision rule \( \delta, d \) can be replaced by \( \delta \).

**3. The BFDR and BFNR.** In this section we will derive general formulas for the BFDR and BFNR of a multiple testing procedure.
The frequentist risk corresponding to $Q_1$, known as the false discovery rate (FDR), is given by

$$FDR = E_{X|\theta} \left[ \sum_{d \in D} \frac{\sum_{i=1}^{n} d_i (1 - h_i)}{\sum_{i=1}^{n} d_i} \delta(d|X) \right]$$

$$= \sum_{I:|I|>0} \left[ \frac{1}{|I|} \sum_{i \in I} (1 - h_i) E_{X|\theta} \left\{ \prod_{i \in I} \delta_i(X) \prod_{i \in I^c} [1 - \delta_i(X)] \right\} \right]$$

$$= \sum_{I:|I|>0} \left[ \frac{1}{|I|} \sum_{i \in I} (1 - h_i) E_{X|\theta} \left\{ \phi_I(X) \right\} \right],$$

where $I \subseteq \{1, \ldots, n\}$ and

$$\phi_I(X) = \prod_{i \in I} \delta_i(X) \prod_{i \in I^c} [1 - \delta_i(X)]$$

is the probability of rejecting the set of null hypotheses $\{H_i, i \in I\}$ and accepting the rest.

Under a prior distribution of $\theta$, the posterior FDR (PFDR) is given by

$$PFDR = E_{\theta|X} \left[ \sum_{d \in D} \frac{\sum_{i=1}^{n} d_i (1 - h_i)}{\sum_{i=1}^{n} d_i} \delta(d|X) \right]$$

$$= \sum_{d \in D} \frac{\sum_{i=1}^{n} d_i r_i(X)}{\sum_{i=1}^{n} d_i} \delta(d|X)$$

$$= \sum_{I:|I|>0} \left[ \frac{1}{|I|} \sum_{i \in I} r_i(X) \phi_I(X) \right],$$

where $r_i(X) = E\{(1 - h_i) | X\} = P\{\theta_i \in \Theta_0 | X\}$, the posterior probability of $H_i$ being true.

The Bayes FDR (BFDR) is the expectation of (9) with respect to $\theta$, or the expectation of (10) with respect to $X$. The BFDR has been referred to as the Average FDR (AFDR) in Chen and Sarkar (2005). Often in the literature, the BFDR is treated as a frequentist FDR under the mixture model represented by the marginal distribution of $X$ (Storey, 2002, 2003; Genovese and Wasserman, 2002; Efron, 2003).

Analogously, the frequentist risk corresponding to $Q_2$, known as the false
non-discovery rate (FNR), is given by:

\[
FNR = E_{X|\theta} \left\{ \sum_{d \in D} \left[ \frac{1}{|I^c|} \sum_{i \in I^c} h_i E_{X|\theta} \{ \phi_I(X) \} \right] \vee 1 \delta(d|X) \right\}
\]

(11)

The posterior FNR (PFNR) is given by

\[
PFNR = E_{\theta|X} \left\{ \sum_{d \in D} \left[ \frac{1}{|I^c|} \sum_{i \in I^c} h_i E_{X|\theta} \{ \phi_I(X) \} \right] \vee 1 \delta(d|X) \right\}
\]

(12)

\[1 - r_i(X) = E \{ h_i | X \} = P \{ \theta_i \in \Theta_i | X \} \]

is the posterior probability of \( H_i \) being false.

The Bayes FNR (BFNR) is the expectation of (11) with respect to \( \theta \), or the expectation of (12) with respect to \( X \).

These FNR-related measures can be equivalently described in terms of quantities that can be interpreted as measures of power in the same spirit as in single testing. For instance, when controlling frequentist FDR is of importance, 1-FNR, which Genovese and Wasserman (2002) call Correct Non-Discovery Rate (CNR), can be considered as a frequentist measure of power. Similar measures can be defined from a Bayesian perspective; for instance, Posterior CNR (PCNR) = 1 - BFNR and Bayes CNR (BCNR) = 1 - BFNR.

Another frequentist concept of power that is being frequently used in multiple testing is the Sensitivity, also known as the Average Power, defined as the expected proportion of false null hypotheses that are rejected, i.e.,

\[
Sensitivity = E_{X|\theta} \left\{ \sum_{d \in D} \left[ \frac{1}{|I^c|} \sum_{i \in I^c} h_i \phi_I(X) \right] \right\}
\]

(13)

\[1 - \sum_{i=1}^{n} h_i > 0. \]

For a Bayesian version of this, we need to properly define the ratio in (13) to incorporate the situation of no false null hypotheses,
which can happen with positive probability. We define this ratio to be 1 when \( \sum_{i=1}^{n} h_i > 0 \). This makes it consistent with the BCNR concept of power. Thus, the Posterior Sensitivity is given by

\[
E_{\theta|X}\left\{ \frac{\sum_{i=1}^{n} h_i \delta_i(X)}{\sum_{i=1}^{n} h_i} \right\} I\left( \sum_{i=1}^{n} h_i > 0 \right) + P_{\theta|X}\left\{ \sum_{i=1}^{n} h_i = 0 \right\}.
\]  

(14)

The Bayes Sensitivity is the expectation of (13) with respect to \( \theta \) or of (14) with respect to \( X \).

4. The BFDR and BFNR of one-step randomized stepwise procedures. We will discuss in this section how the BFDR or BFNR can be written more explicitly for certain randomized stepwise procedures. To that end, let \( r_{1:n}(X) \leq \cdots \leq r_{n:n}(X) \) be the ordered values of \( r_1(X), \ldots, r_n(X) \), and \( (H_i, \delta_i(X)), i = 1, \ldots, n \), be the corresponding pairs of the null hypotheses and their rejection probabilities given \( X \). Let us consider the following type of one-step randomized multiple testing procedure, which is a function of a discrete random variable \( K(X) \) with probability distribution defined on the set \( \{0, 1, \ldots, n\} \). Given \( K(X) = k \), let

\[
\delta_{i:n}(X) = \begin{cases} 
1 & \text{if } i \leq k \\
\delta_{k+1:n}(X) & \text{if } i = k + 1 \\
0 & \text{if } i > k + 1,
\end{cases}
\]  

(15)

with \( \delta_{i:n} = 1 \) \( \forall \) \( i \) if \( K(X) = n \). Let \( \{i_1, \ldots, i_n\} \) be the set of indices such that \( r_{i_j}(X) \equiv r_{j:n}(X), j = 1, \ldots, n \). Then, note that for this procedure, given \( K(X) = k \),

\[
\prod_{i \in I} \delta_i(X) \prod_{i \in I^c} [1 - \delta_i(X)] = \begin{cases} 
1 - \delta_{k+1}(X) & \text{if } I = \{i_1, \ldots, i_k\} \\
\delta_{i_{k+1}}(X) & \text{if } I = \{i_1, \ldots, i_{k+1}\} \\
0 & \text{otherwise.}
\end{cases}
\]  

(16)

Therefore, we have the following:

**Theorem 4.1.** The BFDR of one-step randomized procedure (15) is given
by

\[
\text{BFDR} = E_X \sum_{k=0}^{n} \left\{ \left( 1 - \delta_{k+1:n}(X) \right) A_k(X) + \delta_{k+1:n}(X) A_{k+1}(X) \right\} I(K(X) = k),
\]

(17)

where \( \delta_{n+1:n} = 0 \), \( A_k(X) = \frac{1}{k} \sum_{i=1}^{k} r_{i:n}(X), k = 1, \ldots, n \), and \( A_0 = 0 \).

Consider now a multiple testing procedure (15) with

\[
K_{SD}(X) = \max \{ 1 \leq i \leq n : X_{i:n} \leq c_{i:n} \}
\]

(18)

if the maximum exists, and = 0 otherwise, given some critical values \( c_{1:n} \leq \cdots \leq c_{n:n} \). It is a one-step randomized stepdown procedure in terms of \( X_i \)'s.

The following theorem provides a more explicit expression of the BFDR of this procedure under the iid set-up.

**Theorem 4.2.** Consider testing \( H_i : \theta_i = \theta_0 \) against \( K_i : \theta_i = \theta_1, i = 1, \ldots, n \), for some fixed \( \theta_0 < \theta_1 \). Let \( (X_i, \theta_i), i = 1, \ldots, n \), be iid as \( (X, \theta) \), where \( X \mid \theta \sim f_\theta(x) \) and \( \theta \sim \pi_0 I(\theta = \theta_0) + (1 - \pi_0) I(\theta = \theta_1) \). Assume that \( f_\theta(x) \) is stochastically decreasing in \( \theta \) in the sense that the ratio \( f_\theta'(x) / f_\theta(x) \) is decreasing in \( x \) for any \( \theta < \theta' \). Then, the BFDR of the randomized stepdown procedure \( \delta \) with \( K(X) \) given in (18) and \( \delta_{k+1:n} \) independent of \( X \) is

\[
\text{BFDR} = \pi_0 \sum_{k=1}^{n} \left( 1 - \frac{\delta_{k+1:n}}{k+1} \right) \frac{F_0(c_{k:n})}{F(c_{k:n})} P\{ K_{SD}(X) = k \} + \pi_0 \sum_{k=0}^{n-1} \frac{\delta_{k+1:n}}{k+1} E_X \left\{ \frac{f_0(X_{k+1:n})}{f(X_{k+1:n})} I(K_{SD}(X) = k) \right\},
\]

(19)

where

\[
F_i(x) = P\{ X \leq x \mid \theta = \theta_i \}, i = 0, 1,
\]

\[
F(x) = P\{ X_1 \leq x \} = \pi_0 F_0(x) + (1 - \pi_0) F_1(x),
\]

and \( f_0 \) and \( f \) are the densities of \( F_0 \) and \( F \) respectively.

**Proof.** First note that, since \( r_i(X) = \pi_0 f_0(X_i) / f(X_i) \) is an increasing
function of $X_i$, $r_{i:n}(X) = \pi_0 f_0(X_{i:n})/f(X_{i:n})$. Therefore,

$$E_X \{ A_k(X) I(K_{SD}(X) = k) \} = \pi_0 E_X \left\{ \frac{1}{k} \sum_{i=1}^{k} f_0(X_{i:n}) I(X_{k:n} \leq c_{k:n}, X_{k+1:n} > c_{k+1:n}, \ldots, X_{n:n} > c_{n:n}) \right\}.$$  

Since the $X_i$'s are iid as $f$, conditional on the event $\{X_{k:n} \leq c_{k:n}, X_{k+1:n} > c_{k+1:n}, \ldots, X_{n:n} > c_{n:n}\}$, $X_{1:n} \leq \cdots \leq X_{k:n}$ are the order components of $k$ iid random variables each having the density $f(x)I(x \leq c_{k:n})/F(c_{k:n})$. Hence,

$$E_X \left\{ \frac{1}{k} \sum_{i=1}^{k} f_0(X_{i:n}) \right\} = \int_{-\infty}^{c_{k:n}} \frac{f_0(x)}{f(x)F(c_{k:n})} dx = \frac{F_0(c_{k:n})}{F(c_{k:n})}.$$  

Thus,

$$E_X \{ A_k(X) I(K_{SD}(X) = k) \} = \pi_0 \frac{F_0(c_{k:n})}{F(c_{k:n})} P\{K_{SD}(X) = k\}.$$  

The expression (19) then follows from Theorem 4.1 by noting that

$$A_{k+1}(X) = k \frac{A_k(X)}{k+1} + \frac{1}{k+1} \frac{\pi_0 f_0(X_{k+1:n})}{f(X_{k+1:n})}.$$  

\[ \square \]

**Remark 4.1.** We could have considered in the above theorem $K_i : \theta_i > \theta_0$, $i = 1, \ldots, n$, assumed that $\theta \sim \pi_0 I(\theta = \theta_0) + (1 - \pi_0)\eta(\theta)I(\theta > \theta_0)$, for some probability density $\eta(\theta)$ on $\theta > \theta_0$, and defined $F_1$ as

$$F_1(x) = \int_{\theta_0}^{\infty} P\{X \leq x \mid \theta\} \eta(\theta) d\theta.$$  

When $\delta_{k+1:n} = 0$ in Theorem 4.2; that is, for the non-randomized stepdown procedure with critical values $c_{1:n} \leq \cdots \leq c_{n:n}$, the BFDR simplifies to

$$BFDR = \pi_0 \sum_{k=1}^{n} \frac{F_0(c_{k:n})}{F(c_{k:n})} P\{K_{SD}(X) = k\} = \sum_{k=1}^{n} \frac{n}{k} \frac{F_0(c_{k:n})}{F(c_{k:n})} P\{K_{SD}^*(X) = k - 1\},$$  

(20)
where \( K_{SD}^{*}(X) = \max\{1 \leq i \leq n-1 : X_{i:n-1} \leq c_{i+1:n}\} \), if the maximum exists, and = 0 otherwise, with \( X_{1:n-1} \leq \cdots \leq X_{n-1:n-1} \) being the ordered versions of any \( n-1 \) components of \( X \).

For a non-randomized single-step procedure with the critical value \( c \), the BFDR in Theorem 4.2 simplifies to

\[
\text{BFDR} = \frac{\pi_0 F_0(c)}{F(c)} P\{ K_{SD}(X) > 0 \}. \tag{21}
\]

The first factor in (21), the conditional probability \( P\{h_1 = 0 \mid d_1 = 1\} \), is the Bayesian FDR defined by Efron (2003). It is also the positive FDR (pFDR) due to Storey (2002, 2003) under the iid setup considered in Theorem 4.2.

As seen from the second formula in (20), the BFDR of the non-randomized stepdown procedure with the \( c_{k:n} \)’s satisfying \( F_0(c_{k:n}) = k\alpha/n \) is

\[
\text{BFDR} = \pi_0 \alpha \sum_{k=0}^{n-1} P\{ K_{SD}^{*}(X) = k \} = \pi_0 \alpha. \tag{22}
\]

This is the Benjamini and Hochberg (1995) (BH) procedure. It is not surprising that the BFDR of the BH-procedure under the the above iid set-up is \( \pi_0 \alpha \), as it is known that, conditional on \( \theta \), the FDR of the BH-procedure under this setup is equal to \( p_0 \alpha \), where \( p_0 \) is the proportion of true null hypotheses [Benjamini and Hochberg (1995), Benjamini and Yekutieli (2001), Sarkar (2002) and Finner and Roters (2001)].

The above two results on single-step and the BH procedures have been extended to certain positively dependent distributions in Benjamini and Yekutieli (2001), Sarkar (2002, 2004, 2006).

Analogous to the results on the BFDR, we have the following results related to the BFNR.

**Theorem 4.3.** For a one-step randomized procedure (15), we have

\[
\text{BFNR} = E_X \sum_{k=0}^{n-1} \left\{ \left( 1 - \delta_{k:n}(X) \right) B_{k-1}(X) + \delta_{k:n}(X) B_k(X) \right\} I(K(X) = k), \tag{23}
\]

where \( \delta_{1:n} = 1, B_k(X) = \frac{1}{n-k} \sum_{i=k+1}^{n} (1 - r_{i:n}(X)), k = 0, \ldots, n-1 \) and \( B_n = 0 \).
Theorem 4.4. For the multiple testing problem in Theorem 4.2 and under the model considered therein, the BFNR of the one-step randomized stepup procedure $\delta$ with $\delta_{k:n}$ independent of $X$ and

$$K_{SU}(X) = \min\{1 \leq i \leq n : X_{i:n} \geq d_{i:n}\} - 1$$

if the minimum exists, and $= n$ otherwise, is

$$\text{BFNR} = \pi_1 \sum_{k=0}^{n-1} \left(1 - \frac{1 - \delta_{k:n}}{n-k+1}\right) \frac{\bar{F}_1(d_{k+1:n})}{F(d_{k+1:n})} P\{K_{SU}(X) = k\} +$$

$$\pi_1 \sum_{k=0}^{n} \frac{1 - \delta_{k:n}}{n-k+1} E_X \left\{\frac{f_1(X_{k+1:n})}{f(X_{k+1:n})} I(K_{SU}(X) = k)\right\},$$

(25)

where $\pi_1 = 1 - \pi_0$, $\bar{F}_i = 1 - F_i$, $i = 0, 1$, and $\bar{F} = 1 - F$.

When $\delta_{k:n} = 1$ in Theorem 4.4, that is, for the non-randomized stepup procedure with critical values $d_{1:n} \leq \cdots \leq d_{n:n}$, the BFNR is

$$\text{BFNR} = \pi_1 \sum_{k=0}^{n-1} \bar{F}_1(d_{k+1:n}) \frac{\bar{F}(d_{k+1:n})}{F(d_{k+1:n})} P\{K_{SU}(X) = k\}$$

$$= \pi_1 \sum_{k=0}^{n-1} \frac{n}{n-k} \bar{F}_1(d_{k+1:n}) P\{K_{SU}^*(X) = k\},$$

(26)

where $K_{SU}^*(X) = \min\{1 \leq i \leq n-1 : X_{i:n-1} \geq d_{i+1:n}\}$, if the minimum exists, and $= 0$ otherwise, with $X_{1:n-1} \leq \cdots \leq X_{n-1:n-1}$ being the ordered versions of any $n - 1$ components of $X$.

For the non-randomized single-step procedure with the critical value $d$, we have

$$\text{BFNR} = \pi_1 \frac{\bar{F}_1(d)}{F(d)} P\{K_{SU}(X) < n\}.$$

(27)

And, for the non-randomized stepup procedure with the $d_{k:n}$’s satisfying $\bar{F}(d_{k+1:n}) = (n-k)\beta/n$, $k = 0, \ldots, n-1$, we have

$$\text{BFNR} = \pi_1 \beta \sum_{k=0}^{n-1} P\{K_{SU}^*(X) = k\} = \pi_1 \beta.$$

(28)
The fact that this latter procedure controls the BFNR at $\beta$ also follows from that it controls the frequentist FNR at $\beta$ [Sarkar (2004)].

5. BFDR and BFNR controlling procedures. We now present in this section some procedures that control the BFDR or BFNR. First, we have the following:

**Theorem 5.1.** Let

$$K(X) = \max\{0 \leq j \leq n : A_j(X) \leq \alpha\}. \quad (29)$$

Then, the one-step randomized procedure $\delta$ defined as follows given $K(X) = k$

$$\delta_{i:n}(X) = \begin{cases} 1 & \text{if } i \leq k \\ \frac{\alpha - A_k(X)}{A_{k+1}(X) - A_k(X)} & \text{if } i = k + 1 \\ 0 & \text{otherwise,} \end{cases} \quad (30)$$

with $\delta_{i:n} = 1 \forall i$ when $k = n$, controls the BFDR at $\alpha$.

**Proof.** From Theorem 4.1,

$$\text{BFDR} = E_X \sum_{k=0}^{n-1} \left\{ \left[ (1 - \delta_{k+1:n}(X))A_k(X) + \delta_{k+1:n}(X)A_{k+1}(X) \right] I(K(X) = k) \right\}$$

$$+ E_X \left\{ A_n(X) I(K(X) = n) \right\}$$

$$= \alpha P\{0 \leq K(X) < n\} + E_X\{A_n(X)I(K(X) = n)\}, \quad (31)$$

which is less than or equal to $\alpha$. $\square$

**Remark 5.1.** Clearly, the above procedure does not require any particular dependence structure in the conditional distribution of $X$ given $\theta$. The procedures that control the frequentist FDR, and hence the BFDR, on the other hand, need certain dependence assumptions. For instance, as discussed in the above section, the BFDR of the BH procedure is equal to $\pi_0\alpha$ when $(X_i, \theta_i)$’s are iid, and is less than $\pi_0\alpha$ when $X$, given $\theta$, has some type of positive dependence structure. Without such independence or positive dependence assumptions, the BH procedure may fail to control the BFDR at $\alpha$ [Benjamini and Hochberg (1995); Benjamini and Yekutiel (2001); Sarkar (2002)]. Thus, the above procedure is
applicable in much more general situations. It offers an alternative approach to controlling the BFDR when the BH procedure fails to work. The non-randomized version of the procedure in Theorem 5.1 will also control BFDR, though little more conservatively.

Under the iid set-up considered in Theorem 4.2, the conditional probability $r_i(X)$ simplifies to

$$r_i(X) = \frac{\pi_0 f_0(X_i)}{f(X_i)} = \frac{\pi_0 f_0(X_i)}{\pi_0 f_0(X_i) + (1 - \pi_0) f_1(X_i)}.$$  \hfill (32)

This has been referred to as the local FDR by Efron et al. (2001). Suppose that $f_\theta(x)$ is stochastically decreasing in $\theta$ in the sense that the ratio $f_\theta'(x)/f_\theta(x)$ is decreasing in $x$ for any $\theta < \theta'$. Writing $r_i(X)$ simply as $r(X_i)$, we then see that

$$A_k(X) = \frac{1}{k} \sum_{i=1}^k r(X_{i:n}) \text{ for } k = 1, \ldots, n. \hfill (33)$$

Since

$$\max\{j : A_j(X) \leq \alpha\} \geq \max\{j : r(X_{j:n}) \leq \alpha\}, \hfill (34)$$

the BFDR procedure in Theorem 5.1 under the iid set-up rejects more null hypotheses, and hence more powerful, than the procedure where the PFDR is controlled by rejecting the null hypotheses whose posterior probabilities are less than or equal to $\alpha$. This latter idea was suggested in Efron et al. (2001) in their Bayesian approach to multiple testing.

As discussed in Section 4, alternative procedures controlling the BFDR at $\alpha$ can be obtained. For instance, one can utilize (26) or can use the BH-procedure in terms of the $r(X_i)$'s. Since the BFDR in (26) simplifies to

$$\text{BFDR} = \frac{\pi_0 F_0(c)}{F(c)} \{1 - [1 - F(c)]^n\}, \hfill (35)$$

under the iid set-up, which is increasing in $c$, the BFDR can be controlled at $\alpha$ by using the procedure $\delta$ with

$$K(X) = \max\left\{j : \frac{\pi_0 F_0(X_{j:n})}{F(X_{j:n})} \{1 - [1 - F(X_{j:n})]^n\} \leq \alpha\right\} \hfill (36)$$

if the maximum exists, and $= 0$ otherwise and $\delta_{k+1:n} = 0$. Note that a slightly conservative version of this; that is, the $\delta$ with

$$K(X) = \max\left\{j : \frac{\pi_0 F_0(X_{j:n})}{F(X_{j:n})} \leq \alpha\right\} \hfill (37)$$
if the maximum exists, and $= 0$ otherwise and $\delta_{k+1:n} = 0$, which is the procedure that controls the Bayesian FDR of Efron et al. (2001).

Since $r(X_i)$ is an increasing function of $X_i$, the BH-procedure can be equivalently described in terms of the $r(X_i)$'s; that is, by using the $\delta$ with

$$K(X) = \max\{1 \leq j \leq n : r(X_{j:n}) \leq c_{j:n}\},$$

if the maximum exists, and $= 0$ otherwise and $\delta_{k+1:n} = 0$, where the constants $c_{1:n} \leq \cdots \leq c_{n:n}$ are subject to $P\{r(X_1) \leq c_{j:n} \mid \theta_1 = \theta_0\} = j\alpha/n$.

Notice that a stepup procedure with the same $c_{j:n}$'s; that is, the $\delta$ with

$$K(X) = \min\{1 \leq j \leq n : r(X_{j:n}) \geq c_{j:n}\} - 1$$

if the minimum exists, and $= n$ otherwise and $\delta_{k+1:n} = 0$ also controls the BFDR in the iid case, but more conservatively. This is because, the BFDR of this procedure is

$$\text{BFDR} = E X \sum_{k=1}^{n} \left\{ A_k(X) I(K(X) = k) \right\}$$

and since $A_k$ is increasing in $k$ and the set $\{K(X) \geq k\}$ is smaller compared to that for the $K(X)$ in (42), the summation in (44), and hence its expectation, is smaller than that for the $K(X)$ in (42).

**Remark 5.2.** Under the iid setup in Theorem 4.2, our proposed BFDR procedure is asymptotically (as $n \to \infty$) equivalent to the Benjamini-Hochberg procedure, rejecting all $H_i$ with $X_i \leq X_{K:n}$, where $K$ is defined by

$$K = \max\{1 \leq j \leq n : n \frac{j}{j} \pi_0 F_0(X_{j:n}) \leq \alpha\}.$$

This can be proved as follows:

$$A_j(X) = \frac{\pi_0}{j} \sum_{i=1}^{j} f_0(X_{i:n}) = \frac{\pi_0 n}{j} \frac{1}{n} \sum_{i=1}^{n} f_0(X_i) I(X_i \leq X_{j:n})$$

$$\approx \frac{\pi_0 n}{j} F_0(X_{j:n}),$$

if the maximum exists, and $= 0$ otherwise and $\delta_{k+1:n} = 0$, which is the procedure that controls the Bayesian FDR of Efron et al. (2001).

Since $r(X_i)$ is an increasing function of $X_i$, the BH-procedure can be equivalently described in terms of the $r(X_i)$'s; that is, by using the $\delta$ with

$$K(X) = \max\{1 \leq j \leq n : r(X_{j:n}) \leq c_{j:n}\},$$

if the maximum exists, and $= 0$ otherwise and $\delta_{k+1:n} = 0$, where the constants $c_{1:n} \leq \cdots \leq c_{n:n}$ are subject to $P\{r(X_1) \leq c_{j:n} \mid \theta_1 = \theta_0\} = j\alpha/n$.

Notice that a stepup procedure with the same $c_{j:n}$'s; that is, the $\delta$ with

$$K(X) = \min\{1 \leq j \leq n : r(X_{j:n}) \geq c_{j:n}\} - 1$$

if the minimum exists, and $= n$ otherwise and $\delta_{k+1:n} = 0$ also controls the BFDR in the iid case, but more conservatively. This is because, the BFDR of this procedure is

$$\text{BFDR} = E X \sum_{k=1}^{n} \left\{ A_k(X) I(K(X) = k) \right\}$$

and since $A_k$ is increasing in $k$ and the set $\{K(X) \geq k\}$ is smaller compared to that for the $K(X)$ in (42), the summation in (44), and hence its expectation, is smaller than that for the $K(X)$ in (42).

**Remark 5.2.** Under the iid setup in Theorem 4.2, our proposed BFDR procedure is asymptotically (as $n \to \infty$) equivalent to the Benjamini-Hochberg procedure, rejecting all $H_i$ with $X_i \leq X_{K:n}$, where $K$ is defined by

$$K = \max\{1 \leq j \leq n : n \frac{j}{j} \pi_0 F_0(X_{j:n}) \leq \alpha\}.$$

This can be proved as follows:

$$A_j(X) = \frac{\pi_0}{j} \sum_{i=1}^{j} f_0(X_{i:n}) = \frac{\pi_0 n}{j} \frac{1}{n} \sum_{i=1}^{n} f_0(X_i) I(X_i \leq X_{j:n})$$

$$\approx \frac{\pi_0 n}{j} F_0(X_{j:n}),$$

if the maximum exists, and $= 0$ otherwise and $\delta_{k+1:n} = 0$, which is the procedure that controls the Bayesian FDR of Efron et al. (2001).

Since $r(X_i)$ is an increasing function of $X_i$, the BH-procedure can be equivalently described in terms of the $r(X_i)$'s; that is, by using the $\delta$ with

$$K(X) = \max\{1 \leq j \leq n : r(X_{j:n}) \leq c_{j:n}\},$$

if the maximum exists, and $= 0$ otherwise and $\delta_{k+1:n} = 0$, where the constants $c_{1:n} \leq \cdots \leq c_{n:n}$ are subject to $P\{r(X_1) \leq c_{j:n} \mid \theta_1 = \theta_0\} = j\alpha/n$.

Notice that a stepup procedure with the same $c_{j:n}$'s; that is, the $\delta$ with

$$K(X) = \min\{1 \leq j \leq n : r(X_{j:n}) \geq c_{j:n}\} - 1$$

if the minimum exists, and $= n$ otherwise and $\delta_{k+1:n} = 0$ also controls the BFDR in the iid case, but more conservatively. This is because, the BFDR of this procedure is

$$\text{BFDR} = E X \sum_{k=1}^{n} \left\{ A_k(X) I(K(X) = k) \right\}$$

and since $A_k$ is increasing in $k$ and the set $\{K(X) \geq k\}$ is smaller compared to that for the $K(X)$ in (42), the summation in (44), and hence its expectation, is smaller than that for the $K(X)$ in (42).
since
\[
\frac{1}{n} \sum_{i=1}^{n} \frac{\pi_0 f_0(X_i)}{f(X_i)} I(X_i \leq x) \xrightarrow{P} \pi_0 \mathbb{E} \left\{ \left( \frac{f_0(X_1)}{f(X_1)} \right) I(X_1 \leq x) \right\} = \pi_0 F_0(x).
\]

(43)

Analogous to the result in Theorem 5.1, we also can derive a BFNR-controlling procedure under general dependence conditions.

**Theorem 5.2.** Let
\[
K(X) = \min \{0 \leq j \leq n : B_j(X) \leq \beta\} - 1
\]

(44)

Then, the one-step randomized procedure \( \delta \) defined as follows given \( K(X) = k \)
\[
\delta_{i:n}(X) = \begin{cases} 
1 & \text{if } i < k \\
\frac{B_{k-1}(X) - \beta}{B_{k-1}(X) - B_k(X)} & \text{if } i = k \\
0 & \text{otherwise},
\end{cases}
\]

(45)

with \( \delta_{i:n} = 0 \) \( \forall i \) when \( k = -1 \) controls the BFNR at \( \beta \).

**Remark 5.3.** This BFNR controlling procedure has similar properties of BFDR controlling procedure proposed in Theorem 5.1, that is, it does not depend on any dependence structure.

Alternative procedures controlling the BFNR at \( \beta \) can be obtained as discussed in Section 4. The BFNR in (32) can be simplified as:
\[
BFNR = \frac{\pi_1 \bar{F}_1(d)}{F(d)} \left\{ 1 - [F(d)]^n \right\},
\]

under the iid set-up, which is decreasing in \( d \). The BFNR can be controlled at \( \beta \) by using the \( \delta \) with
\[
K(X) = \min \left\{ j : \frac{\pi_1 \bar{F}_1(X_{j:n})}{F(X_{j:n})} \left\{ 1 - [F(X_{j:n})]^n \right\} \leq \beta \right\} - 1
\]

(47)

if the minimum exists, and \( = n \) otherwise and \( \delta_{k:n} = 1 \). Note that a slightly conservative version of this; that is, \( \delta \) with
\[
K(X) = \min \left\{ j : \frac{\pi_1 \bar{F}_1(X_{j:n})}{F(X_{j:n})} \leq \beta \right\} - 1
\]

(48)
if the minimum exists, and \( n \) otherwise and \( \delta_{k:n} = 1 \), which is the procedure that controls the Bayesian FNR.

Since \( r(X_i) \) is an increasing function of \( X_i \), the FNR procedure by Sarkar (2004) can be equivalently described in terms of the \( r(X_i) \)'s; that is, by using the \( \delta \) with

\[
K(X) = \min\{1 \leq j \leq n : r(X_{j:n}) \geq d_{j:n}\} - 1, \tag{49}
\]

if the maximum exists, and = 0 otherwise and \( \delta_{k:n} = 0 \), where the constants \( d_{1:n} \leq \cdots \leq d_{n:n} \) are subject to \( P\{r(X_1) \geq d_{j:n} \mid \theta_1 = \theta_0\} = (n - j + 1)\beta/n \).

6. Simulations. We now wish to study numerically how our proposed BFDR procedure performs compared to other BFDR procedures, such as the Benjamini-Hochberg procedure and the procedure controlling the Bayesian FDR defined in Efron (2001). Recall from Section 4 [see (22)] that the Benjamini-Hochberg procedure is a non-randomized stepwise procedure, the BFDR of which is exactly \( \pi_0\alpha \) under independence and less than or equal to \( \pi_0\alpha \) under certain types of positive dependence of the test statistics. So, the compatible version of the Benjamini-Hochberg procedure we should be comparing with is the one corresponds to (41). While our procedure will not beat this version of the Benjamini-Hochberg procedure under independence, we expect our procedure to perform much better when there is high dependence, positive or not, among the test statistics. Also, recall that the procedure controlling Efron’s Bayesian FDR is the one discussed following (34) and controls the BFDR conservatively under the iid setup. Whether or not it controls the BFDR under a general dependence situation is not known.

Assuming normal distributions of the test statistics conditional on the parameters, we conduct our simulation studies under three different assumptions about the dependence structure of the test statistics.

- Assumption 1: \( X_i \)'s are independent normal with mean \( \theta_i \) and variance 1:
  \( X_i \mid \theta_i \overset{iid}{\sim} N(\theta_i, 1) \).

- Assumption 2: \( X_i \)'s are multivariate normal with a common positive correlation \( \rho = 0.5 \):
  \( X \mid \theta \sim N_n[\theta; (1 - \rho)I_n + \rho J_n] \).

- Assumption 3: \( X_i \)'s are paired multivariate normal with negative corre-
\[ \rho = -0.5: \]
\[ X|\theta \sim N_n\left[ \theta, I_n \otimes \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right]. \]

One-sided Alternative. Consider testing \( H_i : \theta_i = 0 \) v.s. \( K_i : \theta_i = \delta > 0, \) \( i = 1, \ldots, n, \) and the following prior for the parameters:
\[ \theta_i \overset{iid}{\sim} \pi_0 I(\theta_i = 0) + (1 - \pi_0) I(\theta_i = \delta) \]
for some fixed \( \pi_0. \) With independent \( X_i \)'s, we have
\[ r_i(X) = \frac{\pi_0 \phi(X_i; 0, 1)}{\pi_0 \phi(X_i; 0, 1) + (1 - \pi_0) \phi(X_i; \delta, 1)}, \]
where \( \phi(x; \mu, \sigma^2) \) is the density \( N(\mu, \sigma^2) \) at \( x. \) Notice that \( r_i \) is decreasing in \( X_i \) as \( \delta > 0, \) and hence
\[ A_j(X) = \frac{1}{j} \sum_{i=1}^{j} \pi(X_{n-i+1:n}), \quad j = 1, 2, \ldots, n. \]

When the \( X_i \)'s follow multivariate normal with a common positive correlation \( \rho, \) they can be represented as
\[ X_i = \theta_i + \sqrt{(1 - \rho)} Z_i + \sqrt{\rho} Z_0, \]
in terms of the iid standard normals \( Z_i, \ i = 0, 1, \ldots, n. \) Hence, in this case we have
\[ r_i(X) = \frac{\int_{-\infty}^{\infty} f(X, H_i = 0|z) \phi(z; 0, 1) dz}{\int_{-\infty}^{\infty} f(X|z) \phi(z; 0, 1) dz}, \]
where
\[ f(X|z) = \prod_{j=1}^{n} \{ \pi_0 \phi(X_j; \sqrt{\rho} z, 1 - \rho) + (1 - \pi_0) \phi(X_j; \delta + \sqrt{\rho} z, 1 - \rho) \} \]
and
\[ f(X, H_i = 0|z) = \pi_0 \phi(X_i; \sqrt{\rho} z, 1 - \rho) \times \prod_{j \neq i} \{ \pi_0 \phi(X_j; \sqrt{\rho} z, 1 - \rho) + (1 - \pi_0) \phi(X_j; \delta + \sqrt{\rho} z, 1 - \rho) \}. \]

We simulated both the BFDR and BCNR (1-BFNR) of the BH procedure, Efron’s Bayesian procedure and the proposed BFDR procedure with \( n = 100, \)
\( \delta = 2 \) and \( \pi_0 = (0.25, 0.5, 0.7, 0.8, 0.9, 0.95) \) under each of the above three assumptions regarding the conditional distributions of the \( X_i \)'s. Each simulated value is based on 25,000 replications. The simulated BFDR and BCNR for these three procedures are compared in Figures 1 and 2 respectively under Assumption 1, in Figures 3 and 4 respectively under Assumption 2 and in Figures 5 and 6 respectively under Assumption 3.

Under Assumption 1, all three procedures control the BFDR and there is not much difference in terms of power. Whereas, under Assumption 2, the BH procedure and Efron’s Bayesian procedure both control the BFDR conservatively, while the new BFDR procedure controls it exactly at \( \alpha \). Moreover, in this case, the proposed BFDR procedure is much more powerful than the other two procedures. When Assumption 3 holds, we can see that Efron’s Bayesian procedure and the new BFDR procedure both control the BFDR, but the BH procedure fails to control it when \( \pi_0 \) is close to 1. Also, the new BFDR procedure in this case is much more powerful than the other two procedures.

**Two-sided Alternative.** Consider testing \( H_i : \theta_i = 0 \) v.s. \( K_i : \theta_i \neq 0, \ i = 1, \ldots, n \), and the following prior for the parameters:

\[
\theta_i \overset{iid}{\sim} \pi_0 I(\theta = 0) + (1 - \pi_0) N(0, \tau^2),
\]

for some \( \pi_0 \) and \( \tau^2 \).

Under Assumption 1, we have

\[
r_i(X) = Pr(H_i = 0 | X_i) = \frac{\pi_0 \phi(X_i; 0, 1)}{\pi_0 \phi(X_i; 0, 1) + (1 - \pi_0) \phi(X_i; 0, 1 + \tau^2)},
\]

which is decreasing in \( |X_i| \). Hence,

\[
A_j(X) = \frac{1}{j} \sum_{i=1}^{j} \pi(|X|_{n-i+1:n}), \quad j = 1, 2, \ldots, n.
\]

Under Assumption 2, the conditional probability \( r_i \) is as given in (51), but with

\[
f(X|z) = \prod_{j=1}^{n} \{\pi_0 \phi(X_j; \sqrt{\rho z}, 1 - \rho) + (1 - \pi_0) \phi(X_j; \sqrt{\rho z}, 1 - \rho + \tau^2)\}
\]
and
\[ f(X, H_i = 0|z) = \pi_0 \phi(X_i; \sqrt{\rho} z, 1 - \rho) \times \prod_{j \neq i} \{ \pi_0 \phi(X_j; \sqrt{\rho} z, 1 - \rho) + (1 - \pi_0) \phi(X_j; \sqrt{\rho} z, 1 - \rho + \tau^2) \}. \]

Again, we simulated both the BFDR and BCNR (1-BFNR) of the BH procedure, Efron’s Bayesian procedure and the proposed BFDR procedure with \( n = 100, \tau = (0.5, 1, 4, 10) \) and \( \pi_0 = (0.25, 0.5, 0.7, 0.8, 0.9, 0.95) \) under each of the above three assumptions regarding the conditional distributions of the \( X_i \)'s. This time, each simulated value is based on 10,000 replications. The simulated BFDR and BCNR for these three procedures are now compared in Figures 7 and 8 respectively under Assumption 1, in Figures 9 and 10 respectively under Assumption 2 and in Figures 11 and 12 respectively under Assumption 3.

All three procedures now seem to control the BFDR, although the proposed procedure, as expected, always does so better than the other two under dependence. Interestingly, between the BH and Efron’s procedures, the Efron’s is now seen to be more conservative, particularly when \( \tau \) is large. In terms of the BFNR, there is not much difference among the three procedures.

7. FDR-based variable selection. Motivated by Theorem 5.1, we briefly describe in this section a FDR-based Bayesian variable selection procedure. More specifically, we develop a BFDR-controlling procedure under a model more specific to variable selection and incorporate a Bayesian variable selection procedure [George and McCulloch (1993)] into this framework. We need, however, the full data rather than the test statistics \((X_1, \ldots, X_n)\). Denote the full data as \((Y_i, V_i)\) \(i = 1, \ldots, p\), where \( Y_i \) is a binary random variable and \( V_i \) is an \( n \)-dimensional random vector. We can then consider a hierarchical binary regression model for analysis. At the first stage of the model,
\[ P(Y_i = 1) \overset{\text{ind}}{\sim} \Phi(V_i^T \beta), \]
for some \( \beta = (\beta_1, \ldots, \beta_n) \). For the second stage of the model, we introduce binary-valued latent variables \( \gamma_1, \ldots, \gamma_n \), conditional on which we have,
\[ \beta_i | \gamma_i \sim (1 - \gamma_i)N(0, \tau_i^2) + \gamma_i N(0, \sigma_i^2 \tau_i^2), \]
where \(c_1^2, \ldots, c_p^2\) and \(\tau_1^2, \ldots, \tau_p^2\) are variance components. If \(\gamma_j = 1\), then this indicates that the \(j\)th covariate should be included in the model, while \(\gamma_j = 0\) implies that it should be excluded from the model. The conditional distributions can be easily computed using Gibbs sampling and data augmentation procedures [Albert and Chib (1993)] for calculating the posterior distribution.

Thus, in this framework, rejecting the null hypothesis \(H_i : \gamma_i = 0\) in favor of the corresponding alternative \(K_i : \gamma_i = 1\) using a multiple testing procedure is equivalent to selecting the \(j\)th covariate to include in the model. The Bayesian point of view towards selecting variables requires us to focus on the posterior distributions of \(\gamma_1, \ldots, \gamma_p\). The BFDR-controlling procedure would be based on \(P(\gamma_i = 0 \mid Y), i = 1, \ldots, n\), where \(Y = (Y_1, \ldots, Y_p)\). In particular, Theorem 5.1 motivates the following FDR-based procedure:

(a) Set level to be \(\alpha\).

(b) Find the posterior distribution for the hierarchical regression using Markov Chain Monte Carlo (MCMC) methods.

(c) Based on the MCMC output, calculate the posterior probabilities \(r_i(Y) = P\{\gamma_i = 0 \mid Y\}, i = 1, \ldots, n\), and sort them in increasing order as \(r(1)(Y) \leq \cdots \leq r(n)(Y)\).

(d) Calculate \(A_j(Y) = \frac{1}{j} \sum_{i=1}^j r(i)(Y), j = 1, \ldots, n\).

(e) Find \(K(Y) = \max\{1 \leq j \leq n : A_j(Y) \leq \alpha\}\).

(f) Then, given \(K(Y) = k\), determine the probabilities

\[
\delta(i)(Y) = \begin{cases} 
1 & \text{if } i \leq k \\
\frac{a-A_k(Y)}{A_{k+1}(Y)-A_k(Y)} & \text{if } i = k + 1 \\
0 & \text{otherwise,}
\end{cases}
\]

with \(\delta(i) = 1 \forall i\) if \(k = n\).

(g) Include the variables in the model with the corresponding probabilities in (53).

Notice that the procedure here requires one to average posterior probabilities across the models explored in the MCMC iterations. This will control the BFDR.
at level $\alpha$. Also, one can come up with a BFNR-controlling procedure from Theorem 5.2 based on the MCMC output for $P(\gamma_i = 1|Y) \ i = 1, \ldots, n$.

8. **Conclusion.** In this article, we have developed a general Bayesian procedure for controlling false discovery and nondiscovery rates that allow for arbitrary dependence of the test statistics. The decision theoretic framework allows for exploration of these error rates from both Bayesian and frequentist perspectives.

**References**


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Figure 1: The BFDR in mixture model with one-sided alternatives and $\rho = 0$
Figure 2: The BCNR in mixture model with one-sided alternatives and $\rho = 0$
Figure 3: The BFDR in mixture model with one-sided alternatives and $\rho = 0.5$
Figure 4: The BCNR in mixture model with one-sided alternatives and $\rho = 0.5$
Figure 5: The BFDR in mixture model with one-sided alternatives and $\rho = -0.5$
Figure 6: The BCNR in mixture model with one-sided alternatives and $\rho = -0.5$
Figure 7: The BFDR in mixture model with two-sided alternatives and $\rho = 0$
Figure 8: The BCNR in mixture model with two-sided alternatives and $\rho = 0$
Figure 9: The BFDR in mixture model with two-sided alternatives and $\rho = 0.5$
Figure 10: The BCNR in mixture model with two-sided alternatives and $\rho = 0.5$
Figure 11: The BFDR in mixture model with two-sided alternatives and $\rho = -0.5$
Figure 12: The BCNR in mixture model with two-sided alternatives and $\rho = -0.5$