

Relational and Instrumental Understanding

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This section has been abstracted from the text by Richard Skemp titled “Mathematics in the primary school”. Thus it begins.

This Prologue first appeared in 1976, as an article in the journal Mathematics Teaching. It has since been reproduced three times, and read and discussed in three continents. During this time the terms ‘relational understanding’ and ‘instrumental understanding’ have become part of the language of mathematics education. It is hoped that present readers too will find these ideas a helpful starting point for their own thinking.

Faux amis

Faux amis is a term used by the french to describe words which are the same, or very alike, in two languages, but whose meanings are very different. For example:

French word	Meaning in English
<i>histoire</i>	story, not history
<i>librairie</i>	bookshop, not library
<i>chef</i>	head of an organization, not only chief cook
<i>agrément</i>	pleasure or amusement, not agreement
<i>docteur</i>	doctor (high degree), not medical practioner
<i>médecin</i>	medical practitioner, not medicine
<i>parent</i>	relations in general, including parents

One gets faux amis between English as spoken in different parts of the world. An Englishman asking in America for a biscuit would be given what we call a scone. To get what we call a biscuit, he would have to ask for a cookie. And between English as used in mathematics and in everyday life there are such faux amis as field, group, ring, ideal.

A person who is unaware that the word he is using is a faux ami can make inconvenient mistakes. We expect history to be true, but not a story. We take books without paying from a library, but not from a bookshop; and so on. But in the foregoing examples there are cues which might put one on guard: difference of language, or of country or of context.

If, however, the word is used in the same language, country and context, with two meanings whose difference is as basic as the difference between (say) the meanings of 'history' and 'histoire' (which is a difference between fact and fiction) one may expect serious confusion.

Two such words can be identified in the context of mathematics; and it is the alternative meaning attached to each of these words, by a large following, which in my belief is at the root of many difficulties in mathematics education today.

One of the words is 'understanding'. It was brought to my attention some years ago by Stieg Melin-Olsen, of Bergen University, that there are in current use two meanings of this word. These he distinguishes by calling them 'relational understanding' and 'instrumental understanding'. By the former is meant what I, and probably most readers of this article, have always meant by understanding: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons', without realizing that for many pupils and their teachers the possession of such a rule, and ability to use it, is what they mean by 'understanding'.

Suppose that a teacher remind a class that the area of a rectangle is given by $A = L \times B$. A pupil who has been away says he does not understand, so the teacher gives him an explanation along these lines: 'The formula tells you that to get the area of a rectangle, you multiply the length by breadth'. 'Oh, I see says the child, and gets on with the exercise. If we were now to say to him (in effect), 'You may think you understand, but you don't really,' he would not agree. 'Of course I do. Look, I've got all these answers right.' Nor would he be pleased at our devaluing of his achievement. And within

his meaning of the word, he does understand.

We can all think of examples of this kind: ‘borrowing’ in subtraction, ‘turn it upside down and multiply’ for division by a fraction, ‘take it over to the other side and change sign’ are obvious ones; but once the concept has been formed, other examples of instrumental explanations can be identified in abundance in many widely used texts. Here are two from a text used by a former direct-grant grammar school, now independent, with a high academic standard.

Multiplication of fractions.

To multiply a fraction by a fraction, multiply the two numerators together to make the numerator of the product, and the two denominators to make its denominator.

$$\text{e.g. } \frac{2}{3} \text{ of } \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$$

$$\frac{3}{5} \times \frac{10}{13} = \frac{30}{65} = \frac{6}{13}$$

The multiplication sign \times is generally used instead of the word ‘of’.

Circles.

The circumference of a circle (that is its perimeter, or length of its boundary) is found by measurement to be a little more than 3 times the length of its diameter. In any circle the circumference is approximately 3.1416 times the diameter, which is roughly $3\frac{1}{7}$ times the diameter. Neither of these figures is exact as the number cannot be expressed either as a fraction or a decimal. The number is represented by the Greek letter π (pi).

$$\text{Circumference} = \pi d \text{ or } 2\pi r$$

$$\text{Area} = \pi r^2$$

The reader is urged to try for himself this exercise of looking for and identifying instrumental explanations, both in texts and in classroom. This will have three benefits. (i) For persons like the writer, and most readers of this article, it may provide evidence of what otherwise they would not realize: how widespread is the instrumental approach. (ii) It will help, by repeated examples, to consolidate the two contrasting concepts. (iii) It is a good preparation for trying to formulate the difference in general terms.

If it is accepted that these two categories are both well-filled, by those

pupils and teachers whose goals are respectively relational and instrumental understanding (by the pupil), two questions arise. First, does this matter? And second, is one kind of understanding better than the other? For years I have taken for granted the answers for both these questions: briefly ‘Yes; relational’. But the existence of a large body of experienced teachers and of a large number of texts belonging to the opposite camp has forced me to think more about why I hold this view. In the process of changing the judgement from an intuitive to a reflective one, I think I have learnt something useful. The two questions are not entirely separate, but in the present section I shall concentrate as far as possible on the first: does it matter?

The problem here is that of a mis-match, which arises automatically in any *faux amis* situation, and does not depend on whether A’s or B’s meaning ‘is the right one’. Let us imagine if we can, that school A send a team to school B at a game called ‘football’, but neither team knows that there are two kinds (called ‘association’ and ‘rugby’). School A plays soccer and has never heard of ruggar, and vice versa for B. Each team will rapidly decide that the others are crazy, or a lot of foul players. Team A in particular will think that team B use a mis-shapen ball, and commit one foul after another. Unless the two sides stop and talk about what game they are playing at, the game will break up in disorder and the two teams will never want to meet again.

Though it may be hard to imagine such a situation arising in a football field, this is not a far-fetched analogy for what goes on in many mathematics lessons, even now. There is this play. The encounter is compulsory, on five days a week, for about thirty-six weeks a year, over ten years or more of a child’s life.

Leaving aside for the moment whether one kind of understanding is better than the other, there are two kinds of mathematical mis-match that can occur.

1. Pupils whose goal is to understand instrumentally, taught by a teacher who wants them to understand relationally.
2. The other way about

The first of these will cause fewer problems short-term to the pupils, though it will be frustrating for the teacher. The pupils just ‘won’t want to know’ all the careful groundwork he gives in preparation for whatever is to

be learnt next, not his careful explanation. All they want is some kind of rule for getting the answer. As soon as this is reached, they latch on to it and ignore the rest.

If the teacher asks a question that does not quite fit the rule, of course they will get it wrong. For the following example I have to thank Mr. Peter Burney, at that time a student a Coventry College of Education on teaching practice. While teaching area he became suspicious that the children did not really understand what they were doing. So he asked them: 'What is the area of a field 20 cms by 15 yards?' The reply was: '300 square centimetres.' He asked them 'Why not 300 square yards?' Answer: 'Because area is always in square centimetres.'

To prevent errors like the above the pupils need another rule (or of course, relational understanding), that both dimensions must be in the same unit. This anticipates one of the arguments which I shall use against instrumental understanding, that is usually involves a multiplicity of rules rather than fewer principles of more general application.

There is of course always the chance that a few of the pupils will catch on to what the teacher is trying to do. If only for the sake of these, the effort should be maintained. But by being able to use the rule is not enough will not be well received. 'Well is the enemy of better', and if pupils can get the right answer by the kind of thinking they are used to, they will not take kindly to suggestions that they should try for something beyond this.

The other mis-match in which pupils are trying to understand relationally but the teaching makes this impossible, can be a more damaging one. An instance which stays in my memory of a neighbour's child, then seven years old. He was a very bright little boy, with an I.Q. of 140. At the age of five he could read *TheTimes*, but at seven he regularly cried over his mathematics homework. His misfortune was that he was trying to understand relationally teaching which could not be understood in this way. My evidence for this belief is that when I taught him relationally myself, with the help of Unifix, he caught on quickly and with real pleasure.

A less obvious mis-match is that which may occur between teacher and text. Suppose that we have a teacher whose conception of understanding is instrumental, who for one reason or other is using a text which aims for relational understanding by the pupil. It will take more than this to change his teaching style. I was in a school which was using my own text, and noticed

(they were at chapter 1) that some of the pupils were writing answers like:

‘the set of {flowers}’

When I mentioned this to the teacher (he was the head of mathematics) he asked the class to pay attention to him and said: ‘Some of you are not writing your answers properly. Look at the example in the book, at the beginning of the exercise, and be sure to write your answers like that.’

Much of what is being taught under the description of ‘modern mathematics’ is being taught and learnt as instrumentally as were the syllabi which have been replaced. This is predictable from the difficulty we normally experience in accomodating (restructuring) our existing schemas. To the extent that this is so, the innovations have probably done more harm than good, by introducing a mis-match between the teacher and the aims implicit in the new content. For the purpose of introducing ideas such as sets, mappings and variables is the help which, rightly used, they can give to relational understanding. If pupils are still being taught instrumentally, then a ‘traditional’ syllabus will probably benefit them more. They will at least acquire proficiency in a number of mathematical techniques which will be of use to them in other subjects, and the lack of which has recently been a subject of complaints by teachers of science, employers and others.

Near the beginning of this chapter I said that two major *faux amis* could be identified in the context of mathematics. The second one is even more troublesome; it is the word ‘mathematics’ itself. For we are not talking about better and worse teaching of the same kind of mathematics. It is easy to think this, just as our imaginary soccer players who do not know that their opponent are playing a different game might think that the other side pick up the ball and run with it because they cannot kick properly, especially with such mis-shapen ball— in which case they might kindly offer them a better ball and some lessons in dribbling. It has taken me some time to realize that this is not the case.

I used to think that maths teachers are all teaching the same subject, some doing it better than others. I now believe that *there are two effectively different subjects being taught under the same name, ‘mathematics’*. If this is true, then this difference matters beyond any of the differences in syllabi and which are so widely debated. So I would like to try to emphasize the point with the help of another analogy.

Imagine two groups of children are taught music as a pencil-and-paper subject. They are all shown the five line stave, with the curly ‘treble’ sign at the beginning; and taught that the marks on the line are called E, G, B, D, F. Marks between the lines are called F, A, C, E. They learn that a mark with an open oval is called a minim, and is worth two marks with blacked-in ovals which are called crotchets, or four with blacked-in ovals and a tail which are called quavers, and so on- musical multiplication tables if you like. For one group of children, all their learning is of this kind and nothing beyond. If they have a music lesson a day, five days a week in school terms, and are told that it is important, these children could in time probably learn to write out the marks for simple melodies such as those for ‘God Save the Queen’, and ‘Auld Lang Syne’, and to solve simple problems such as ‘What time is this in?’ and ‘What Key?’, and even ‘Transpose this melody from C major to G major’. They would find it boring, and the rules to be memorized would be numerous that problems like ‘Write a simple accompaniment for this melody’ would be too difficult for most. They would give up the subject as soon as possible, and remember it with dislike.

The other group is taught to associate certain sounds with the marks on the paper. For the first few years these are audible sounds, which they make on simple instruments. After a time they can still imagine the sounds whenever they see or write the marks on the paper. Associated with every linear sequence of marks is a melody, and with every vertical set a harmony. The keys C major and G major have an audible relationship, and a similar relationship can be found between certain other pairs of keys. And so on. Much less memory work is involved, and what has to be remembered is largely in the form of related wholes (such as melodies) which their minds easily retain. Exercises such as were mentioned earlier (‘Write a simple accompaniment’) would be within the ability of most. These children would also find their learning intrinsically pleasurable, and many would continue it voluntarily, even after O-level or C.S.E.

For the present purpose I have invented two non-existent kinds of ‘music lesson’, both pencil-and-paper exercises (in the second case, after the first year or two). But the difference between these imaginary activities is no greater than that between two activities which go under the name of mathematics. (We can make the analogy closer, if we imagine that the first group of children were initially taught sounds for the notes in a rather half-hearted way, but that the associations were too ill-formed and unorganized to last.)

The above analogy is, clearly, heavily biased in favor of relational math-

ematics. This reflects my own point of view. To call it a point of view, however, implies that I no longer regard it as a self-evident truth which requires no justification: which it can hardly be if many experienced teachers continue to teach instrumental mathematics. The next step is to try to argue merits of both points of view as clearly and as fairly as possible; and especially of the point of view opposite to one's own. This is why the next section is called 'Devil's advocate'.

Devil's advocate

Given that so many teachers teach instrumental mathematics, might this be because it does have certain advantage? I have been able to think of three advantages (as distinct from situational reasons for teaching this way, which will be discussed later).

1. Within its own context, *instrumental mathematics is usually easier to understand*; sometimes much easier. Some topics such as multiplying two negative numbers together, or dividing by a fractional number, are difficult to understand relationally. 'Minus times minus equals plus', and 'To divide by a fraction you turn it upside down and multiply' are easily remembered rules. If what is wanted is a page of right answers, instrumental mathematics can provide this more quickly and easily.
2. So *the rewards are more immediate, and more apparent*. It is nice to get a page of right answers, and we must not under-rate the importance of the feeling of success which pupils get from this. Recently I visited a school where more of the students described themselves as 'thickos'. Their teachers used the terms too. These children need success to restore their self-confidence, and it can be argued that they can achieve this more quickly and easily in instrumental mathematics than in relational.
3. Just because less knowledge is involved, *one can often get the right answer more quickly and reliably* by instrumental thinking than relational. The difference is so marked that even relational mathematicians use instrumental thinking. This is a point of much interest, which I hope to discuss more fully on a future occasion.

The above may well not do full justice to instrumental mathematics. I shall be glad to know of any further advantages which it may have.

The case for relational mathematics

There are four advantages (at least) in relational mathematics.

1. *It is more adaptable to new tasks.* Recently I was trying to help a boy who had learnt to multiply two decimal fractions together by dropping the decimal point, multiplying as for whole numbers, and re-inserting the decimal point to give the same total number of digits after the decimal point as there were before. This is a handy method if you know how it works. Though no fault of his own, this child did not; and not unreasonably, applied it to the division of decimals. By this method $4.8 \div 0.6$ came to 0.08. The same pupil had also learnt that if you know two angles of a triangle, you can find the third by adding the two given angles together and subtracting from 180° . He got ten questions right this way (his teacher believed in plenty of practice), and went on to use the same method for finding the exterior angles. So he got the next five answers wrong.

I do not think he was being stupid in either of these cases. He was simply extrapolating from what he already knew. But relational understanding, by knowing not only what method worked but why, would have enabled him to relate to new problems. Instrumental mathematics necessitate memorizing which problems a method works for and which not, and also learning a different method for each new class of relational mathematics.

2. *It is easier to remember.* There is a seeming paradox here, in that it is certainly harder to learn. It is certainly easier for people to learn that ‘area of the triangle = $\frac{1}{2}$ base \times height’ than to learn why this is so. But they have to learn separate rules for triangles, rectangles, parallelograms, trapeziums; whereas relational understanding consists partly in seeing all of these in relation to the area of a rectangle. It is still desirable to know the separate rules; one does not have to derive them afresh every time. But knowing also how they are inter-related enables one to remember them as parts of the whole connected, which is easier. There is more to learn—the connections as well as the separate rules—but the result, once learnt is more lasting. So there is less re-learning to do, and long-term the time taken may well be less altogether. Teaching for relational understanding may also involve more actual content. Earlier, an instrumental explanation was quoted leading to the statement ‘Circumference = πd ’. For relational understanding of this, idea of a proportion would have to be taught first (among others), and

this would make it much longer job than teaching the rules as given. But proportionality has such a wide range of other applications that it is worth teaching on these grounds also. In relational mathematics this happens rather often. Ideas required for understanding a particular topic turn out to be basic for the understanding of many other topics too. Sets, mapping, and equivalence are such ideas. Unfortunately their potential benefits are often lost by teaching them as separate topics, rather than as fundamental concepts by which the whole areas of mathematics can be inter-related.

3. *Relational knowledge can be effective as a goal in itself.* This is an empiric fact, based on evidence from controlled experiments using non-mathematical material. The need for external rewards and punishments is greatly reduced, making what is often called the ‘motivational’ side of a teacher’s job much easier. This is related to:
4. *Relational schemas are organic in quality.* This is the best way I have been able to formulate a quality by which they seem to act as agents of their own growth. The connection with 3 is that if people get satisfaction from relational understanding, they may not only try to understand relationally new material which is put before them, but also actively seek out new material and explore new areas, very much like a tree extending its roots or an animal exploring new territory in search of nourishment. To develop this idea beyond the level of an analogy is beyond the scope of the present paper, but it is too important to leave out.

If the above is anything like a fair presentation of the arguments for each side, it would appear that a case might exist for instrumental mathematics short-term and with a limited context, long-term in the context of a child’s whole education it does not. So why are so many children taught only instrumental mathematics throughout their school careers? Unless we can answer this, there is little hope of improving the situation.

An individual teacher might take a reasoned choice to teach for instrumental understanding on one or more of the following grounds.

1. That relational understanding would take too long to achieve, and to be able to use a particular technique is all that these pupils are likely to need.

2. That relational understanding of a particular topic is too difficult, but the pupils still need to learn the topic for examination reasons.
3. That a skill is needed for use in another subject (e.g. science) before it can be understood relationally with schemas currently available with pupils.
4. That he is a junior teacher in a school where all the other mathematics teaching is instrumental.

All of these imply, as does the phrase ‘make a reasoned choice’ that he is able to consider the alternative goals of instrumental and relational understanding on their merits and in relation to a particular situation. To make an informed choice of this kind implies awareness of the distinction, and the relational understanding of the mathematics itself. So nothing else but relational understanding can be adequate for a teacher. One has to face the fact that this is absent in many who teach mathematics; perhaps even a majority.

Situational factors that contribute to the difficulty include:

1. *The backwash effect of examinations.* In view of the importance of examinations for future employment, one can hardly blame pupils if success in these is one of their major aims. The way pupils work cannot but be influenced by the goal for which they are working, which is to answer correctly a sufficient number of questions.
2. *Over-burdened syllabi.* Part of the trouble here is the high concentration of information content of mathematics. A mathematical statement may condense into a single line as much as in another subject might take over one or two paragraphs. By mathematicians accustomed to handling such concentrated ideas, this is often overlooked (which may be why most mathematics lecturers go too fast). Non-mathematicians do not realize it at all. Whatever the reason, almost all syllabi would be much better if reduced in amount so that there would be time to teach them better.
3. *Difficulty in assessment of whether a person understands relationally or instrumentally.* For the marks he makes on paper, it is very hard to make valid inference about the mental process by which a pupil has been led to make them; hence the difficulty of sound examining in mathematics. In a teaching situation, talking with the pupil is almost certainly the best way to find out; but in a class of over thirty, it may be difficult to find the time.

4. *The great psychological difficulty for teachers of accomodating (restructuring) their existing and long-standing schemas, even for the minority who knows the need to, want to do so, and have time for study.*

From a recent article discussing the practical, intellectual and cultural value of mathematics education (and I have no doubt that he means relational mathematics!) by Sir Hermann Bondi, I take these three paragraphs. (In the original, they are not consecutive).

So far my glowing tribute to mathematics has left out a vital point: the rejection of mathematics by so many, a rejection that in not a few cases turns to abject fright.

The negative attitude to mathematics, unhappily so common, even amongst otherwise highly-educated people is surely the greatest measure of our failure and a real danger to our society.

This is perhaps the clearest indication that something is wrong, and indeed very wrong, with the situation. It is not hard to blame education for at least a share of the responsibility; it is harder to pinpoint the blame, and even more difficult to suggest remedies.

If for ‘blame’ we may substitute ‘cause’, there can be small doubt that the widespread failure to teach relational mathematics— a failure to be found in primary, secondary and further education, and in ‘modern’ as well as ‘traditional’ courses can be identified as a major cause. To suggest new remedies is indeed difficult, but it may be hoped that diagnosis is one good step towards a cure. Another step will be offered in the next section.

A theoretical formulation

There is nothing so powerful for directing one’s action in a complex situation, and for co-ordinating one’s own effort with those of others, as a good theory. All good teachers build up their own stores of empirical knowledge, and have abstracted from these some general principles on which they rely for guidance. But while their knowledge remains in this form it is largely still at the intuitive level within individuals, and cannot be communicated, both for this reason and because there is no shared conceptual structure (schema) in terms of which it can be formulated. Were this possible, individual efforts could be integrated into a unified body of knowledge which would be available for use by newcomers to the profession. At present most teachers have to learn from their own mistakes.

For some time my own comprehension of the difference between the two kinds of learning which lead respectively to relational and instrumental mathematics remained at the intuitive level, though I was personally convinced that the difference was one of great importance, and this view was shared by most of those with whom I discussed it. Awareness of the need for an explicit formulation was forced on me in the course of two parallel research projects; and insight came, quite suddenly during a recent conference. Once seen it appears quite simple, and one wonders why I did not think of it before. But there are two kinds of simplicity: that of naivety; and that which by penetrating beyond superficial differences, brings simplicity by unifying. It is the second kind which a good theory has to offer, and this is harder to achieve.

A concrete example is necessary to begin with. When I went to stay in a certain town for the first time, I quickly learnt several particular routes. I learnt to get between where I was staying and the office of a colleague with whom I was working; between where I was staying and the university refectory where I ate; between my friend's office and the refectory; and two or three others. In brief, I learnt a limited number of fixed plans by which I could get from particular starting locations to particular goal locations.

As soon as I had some free time I began to explore the town. Now I was not wanting to get anywhere specific, but to learn my way around, and in the process to what I might come upon that was of interest. At this stage my goal was different one: to construct in my map a cognitive map of the town.

These two activities are quite different. Nevertheless they are, to an outside observer, difficult to distinguish. Anyone seeing me walk from A to B would have great difficulty in knowing (without asking me) which of the two I was engaged in. But the most important thing is about the activity is its goal. In one case my goal was to get to B, which is a physical location. In the other it was to enlarge or consolidate my mental map of the town, which is a state of knowledge.

A person with a set of fixed plans can find his way from a certain starting point to a certain set of goals. The characteristic of a plan is that it tells him what to do at each choice point: turn right out of the door, go straight on past the church, and so on. But if at any stage he makes a mistake, he will be lost; and he will stay lost if he is not able to retrace his steps and get back on the right path.

In contrast, a person with a mental map of the town has something from which he can produce, when needed, an almost infinite number of plans by which he can guide his steps from any starting point to any finishing point, provided only that both can be imagined on his mental map. And if he does take a wrong turn, he will still know where he is, and thereby be able to correct his mistakes without getting lost; even perhaps to learn from it.

The analogy between the foregoing and the learning of mathematics is close. The kind of learning which leads to instrumental mathematics consists of learning of an increasing number of fixed plans, by which pupils can find their way from particular starting points (the data) to required finishing points (the answer to the questions). The plan tells them what to do at each choice point. as in the concrete example, *what has to be done next is determined purely by local situation*. ('When you see the post office, turn left.' 'When you have cleared brackets, collect like terms.') There is no awareness of the overall relationship between successive stages, and the final goal. And in both cases, the learner is dependent on outside guidance of learning each new 'way to get there'.

In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point. (I say 'in principle' because of course some of these paths will be much harder to construct than others.)

This kind of learning is different in several ways from instrumental learning.

1. The means become independent of particular ends to be reached there by.
2. Building up a schema within a given area of knowledge becomes an intrinsically satisfying goal in itself.
3. The more complete a pupil's schema, the greater his feeling of confidence in his own ability to find new ways of 'getting there' without outside help.
4. But a schema is never complete. As our schemas enlarge, so our awareness of possibilities is thereby enlarged. Thus the process often becomes self-continuing, and (by virtue of 3) self-rewarding.

Taking again for a moment the role of the devil's advocate, it is fair to ask whether we are indeed talking about two subjects, relational mathematics and instrumental mathematics, or just two ways of thinking about the same subject matter. Using my earlier analogy, the processes described might be regarded as two different ways of knowing about the same town; in which case the distinction made between relational and instrumental understanding would be valid, but not that between instrumental and relational mathematics.

But what constitutes mathematics is not the subject matter, but a particular kind of knowledge about it. The subject matter of relational and instrumental mathematics may be the same: cars travelling at uniform speeds between two towns whose heights are to be found, bodies falling freely under gravity, etc. etc. But the two kinds of knowledge are so different that there is strong case for regarding them as different kind of mathematics. If this distinction is accepted, then the word 'mathematics' is for many children indeed a false friend, as they find to their cost.