**Nonparametric Methods**

Testing with Ordinal Data or Nonnormal Distributions

Chapter 16

**Rational**

The classical inference methods require a distribution to describe the population of a variable. Only the parameters of the assumed distribution are unknown. They are estimated and understood through testing of hypotheses etc. Such methods are called parametric. The large sample provides theoretical justification for the parametric methods even when the assumption of a specific distribution is not wholly correct. If it is desired to relax the assumption of a parametric distribution and make inference from a moderate size sample only, then nonparametric methods are appropriate. They use ranks or the counts rather than actual values of a random sample observed for a variable.

**Objectives**

To understand

- Sign test for testing hypotheses about the median
- Sign test for comparing paired samples
- Wilcoxon rank-sum (Mann-Whitney) test for comparing unpaired samples

**Outlines**

Testing hypotheses about the median
Comparing two paired samples
Comparing two unpaired samples
Testing Hypotheses about the Median (Wilcoxon Sign Test)

If the population is skew, the median, $\theta$, is an appropriate measure of its center. Since median is the 50th percentile, the number of observations in a random sample of size $n$ below the median would follow a binomial distribution with parameters $\pi = 0.5$. The sign test makes use of this fact to test hypotheses about the median. For example, consider the following hypotheses about the median.

$$H_0: \theta = \theta_o \text{ versus } H_a: \theta \neq \theta_o$$

These can be equivalently stated as,

$$H_0: \pi = 0.5 \text{ versus } H_a: \pi \neq 0.5.$$  

Let $X$ be the number of sample items less than $\theta_o$. Reduce the sample size $n$ by the number sample values equal to $\theta_o$. Let the modified sample size be $m$.

Compute,

$$Z = \frac{U - m}{\sqrt{m}}.$$  

At $\alpha$ level of significance, reject $H_o$ if $Z$ is either greater than $Z_{\alpha/2}$ or less than $-Z_{\alpha/2}$, where $Z_{\alpha/2}$ is $(1-\alpha/2)$th percentile of the standard normal distribution. If $\alpha = .05$, then $(1-.025)100^{th} = 97.5^{th}$ percentile of the standard normal distribution is 1.96. This decision rules serve well if $n$ is large. For small $n$, an exact test can be carried as outlined in 3 steps described below.

- Count the number of sample values, $U$, that are less than $\theta_o$.
- Reduce the sample size $n$ by the number sample values equal to $\theta_o$. Let the modified sample size be $m$.
- Reject $H_o$ at a specified level of significance, if $U$ falls outside the limits found from a table for sign test corresponding to the modified sample size $m$.

Illustrate by Example in Section 16.1

Testing for difference in paired samples (Wilcoxon Sign Test)

We can have a nonparametric test, parallel to the paired t-test, for testing the equality of two medians for two paired populations. Let $X$ and $Y$ denotes variables to measure the same aspect for units of a population before and after they have been subjected to a treatment. If $\theta_1$ and $\theta_2$ are their medians, one might consider testing

$$H_0: \theta_1 = \theta_2 \text{ versus } H_a: \theta_1 \neq \theta_2.$$  

If $H_0$ is true then $\pi = P(X < Y) = P(X > Y) = 0.5$. Hence once again, above hypotheses can equivalently be stated as,

$$H_0: \pi = 0.5 \text{ versus } H_a: \pi \neq 0.5.$$
For testing, a random sample of $n$ units will be selected from the population. On each selected unit, the $(X, Y)$ pair is observed. Thus $(X_i, Y_i)$, $i = 1, 2, \ldots, n$, is a random sample of $n$ pairs. Define,

$$U = \text{number of pairs } (X_i, Y_i) \text{ such that } X_i < Y_i.$$ 

Obviously, $U$ has a binomial distribution with $\pi = 0.5$ if $H_0$ is true and $\pi \neq 0.5$ if $H_o$ is not true. Now the $Z$ statistic as defined above or the sign test as described above can be used.

Illustrate by Examples in Section 6.2

**Comparing two unpaired samples (Mann-Whitney Test)**

There is a nonparametric test parallel to comparing the means of two independent populations. Since populations may not be normal, the nonparametric test compares the medians $\theta_1$ and $\theta_2$.

$$H_0: \theta_1 = \theta_2 \text{ versus } H_1: \theta_1 \neq \theta_2.$$ 

In fact the two sample nonparametric test is valid for more general hypotheses comparing two populations.

$$H_0: \text{The two samples came from the same population}$$
$$H_1: \text{The two samples came from different populations}$$

Two random samples $X_{i1}, i=1, 2, \ldots, n_1$, and $Y_{i2}, i=1, 2, \ldots, n_2$ are taken; one from each population. An assumption is that the samples are independent random samples from continuous distributions, with the distributions having the same shape. The test consists of the following steps.

1. Put both samples in the same column and rank them from 1 through $n = n_1 + n_2$.
2. Find the average rank for each sample, $\bar{R}_1$ and $\bar{R}_2$.
3. Compute the SE of $\bar{R}_1 - \bar{R}_2$. That is, $S_{\bar{R}_1-\bar{R}_2} = \sqrt{\frac{n+1}{12n_1n_2}}$.
4. Compute the test statistic. That is $Z = \frac{\bar{R}_1 - \bar{R}_2}{S_{\bar{R}_1-\bar{R}_2}}$.
5. At $\alpha$ level of significance reject if absolute value of $Z$ is greater than $(1 - \frac{\alpha}{2})100^{th}$ percentile of the standard normal distribution. If $\alpha = .05$, then $(1-.025)100^{th} = 97.5^{th}$ percentile of the standard normal distribution is 1.96.

*Illustrate by Examples in Section 16.3.*
Comparing more than two samples (Kruskal-Wallis Test)

We can perform a test by Kruskal-Wallis to compare medians for two or more populations. This test is a generalization of the Mann-Whitney test and offers a nonparametric alternative to the one-way analysis of variance. The hypotheses are:

\[ H_0: \text{the population medians are all equal} \]
\[ H_a: \text{the medians are not all equal} \]

In fact the Kruskal-Wallis test may be used for general hypotheses comparing populations rather than their medians only.

\[ H_0: \text{The samples came from the same population} \]
\[ H_a: \text{The samples came from at least two different populations} \]

An assumption for this test is that the samples from the different populations are independent random samples from continuous distributions, with the distributions having the same shape.

I illustrate the Kruskal-Wallis test data from Table 15.1.1 and Table 15.5.3 of Chapter 15.