Chapter 6. Inner Product Spaces

• So far our primary use of vector spaces and subspaces has been as algebraic structures.

• For instance, the algebra of linear combinations of vectors was used to generate solutions of a homogeneous linear system $Ax = 0$. We also used bases of eigenvectors to develop approximations and compress information contained in a matrix.

• However, using the dot product in $\mathbb{R}^n$ we saw that we could determine the length of a vector and, at least for $\mathbb{R}^2$ and $\mathbb{R}^3$, we developed the notion of an angle between vectors. These concepts hinted at geometric aspects of vector spaces and subspaces.
Section 6.1 The Dot Product in $\mathbb{R}^n$ and $\mathbb{C}^n$

The dot product of a pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is

$$\mathbf{x} \cdot \mathbf{y} = [x_1 \ x_2 \ \cdots \ x_n] \cdot [y_1 \ y_2 \ \cdots \ y_n]$$

$$= x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{j=1}^{n} x_jy_j.$$  

We refer to (1) as the **standard dot product** on $\mathbb{R}^n$.

The **complex dot product** of a pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{C}^n$ is

$$\overline{\mathbf{x}} \cdot \mathbf{y} = \sum_{j=1}^{n} \overline{x_j}y_j$$

where the bar indicates the conjugate.

It is convenient at this time to introduce a notation for dot products that can be used for $\mathbb{R}^n$ or $\mathbb{C}^n$, where we let the nature of the vectors determine whether to use the computational steps in expressions.

For vectors $\mathbf{x}$ and $\mathbf{y}$ let $(\mathbf{x}, \mathbf{y})$ denote the dot product.
Example 1.

a) Let \( x = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} \) and \( y = \begin{bmatrix} 2 \\ -2 \\ 5 \\ 8 \end{bmatrix} \). Then from (1) we have

\[
(x,y) = (3)(2) + (2)(-2) + (-1)(5) + (0)(8) = 6 - 4 - 5 + 0 = -3.
\]

b) Let \( v = \begin{bmatrix} 2 - i \\ 2i \\ 4 + 3i \end{bmatrix} \) and \( w = \begin{bmatrix} 1 + 2i \\ 3 - 2i \\ 2 \end{bmatrix} \). Then from (2) we have

\[
(v,w) = (2 - i)(1 + 2i) + (2i)(3 - 2i) + (4 + 3i)(2) = 4 - 7i.
\]
The **length** of a vector $\mathbf{x}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is denoted by $\|\mathbf{x}\|$ and is determined from the expression

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})} = \sqrt{\sum_{j=1}^{n} x_j^2}.$$ 

We say that a pair of vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** provided $(\mathbf{x}, \mathbf{y}) = 0$.

**IMPORTANT RESULT:**

Show that the set of all vectors orthogonal to a vector $\mathbf{v}$ is a subspace.
The dot product gives us a tool with which we can define the distance between vectors. For a pair of vectors \( \mathbf{x} \) and \( \mathbf{y} \) in \( \mathbb{R}^n \) or \( \mathbb{C}^n \), the distance between the two vectors is given by

\[
D(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \|.
\]

**Example** Let \( \mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} \) and \( \mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 5 \\ 8 \end{bmatrix} \). Then

\[
D(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \| = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -6 \\ -8 \end{bmatrix} = \sqrt{101}.
\]

**Terminology**

<table>
<thead>
<tr>
<th>Dot product in ( \mathbb{R}^n ) and ( \mathbb{C}^n ).</th>
<th>The length or norm of a vector.</th>
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<tbody>
<tr>
<td>Unit vector.</td>
<td>Orthogonal vectors.</td>
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<tr>
<td>The distance between vectors.</td>
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