Mathematics has been called the science of patterns.

The identification of patterns and common features in seemingly diverse situations provides us with opportunities to unify information.

This approach can lead to the development of classification schemes and structures, which can be studied independently of a particular setting or application.

Thus we can develop ideas that apply to each and every member of the class based upon properties that they have in common. The result is called an abstract model or abstract structure. In many ways this can help us work with more difficult and comprehensive ideas.

Linear algebra uses extensively a structure called vector space. The properties of a vector space give the study of matrices a more geometrical foundation. The corresponding geometrical viewpoint on topics aids both in understanding as well as in suggesting new approaches.
Definition  Let $V$ be a set with two operations $\oplus$, and $\odot$. $V$ is called a real vector space, denoted $(V, \oplus, \odot)$, if the following properties or axioms hold:

1. If $v$ and $w$ belong to $V$, then so does $v \oplus w$. (We say $V$ is closed with respect to operation $\oplus$.)
   a) $v \oplus w = w \oplus v$, for all $v$ and $w$ in $V$. (Commutativity of $\oplus$.)
   b) $(v \oplus w) \oplus u = v \oplus (w \oplus u)$ for all $u$, $v$, and $w$ in $V$. (Associativity of $\oplus$.)
   c) There exists a unique member $z$ in $V$, such that $v \oplus z = z \oplus v = v$, for any $v$ in $V$. ($z$ is called the identity or zero vector for $\oplus$.)
   d) For each $v$ in $V$ there exists a unique member $w$ in $V$ such that $v \oplus w = w \oplus v = z$. ($w$ is denoted $-v$ and called the negative of $v$.)

2. If $v$ is any member of $V$ and $k$ is any real number, then $k \odot v$ is a member of $V$. (We say $V$ is closed with respect to $\odot$.)
   e) $k \odot (v \oplus w) = (k \odot v) \oplus (k \odot w)$ for any $v$ and $w$ in $V$ and any real number $k$. (Distributivity of $\odot$ over $\oplus$.)
   f) $(k + j) \odot v = (k \odot v) \oplus (j \odot v)$ for any $v$ in $V$ and any real numbers $j$ and $k$. (Distributivity of addition of real numbers over $\odot$.)
   g) $k \odot (j \odot v) = (kj) \odot v$, for any $v$ in $V$ and any real numbers $j$ and $k$.
   h) $1 \odot v = v$, for any $v$ in $V$. 

The following terminology is used.

- The members of V are called vectors and need not be directed line segments or row or column vectors. Consider the term vector as a generic name for a member of a vector space.

- The real numbers are called scalars.

- The operation $\oplus$ is called vector addition. The actual definition of this operation need not resemble ordinary addition. The term vector addition is a generic name for an operation that combines two vectors from V and produces another vector of V.

- The operation $\odot$ is called scalar multiplication. It combines a real scalar and a vector from V to produce another vector in V.

- If this definition holds with real numbers replaced by complex numbers, then the structure $(V, \oplus, \odot)$ is called a complex vector space. In this case the complex numbers are called scalars.

The definition of a vector space looks complicated because V, $\oplus$, and $\odot$ have not been given specific meanings. In a particular example the members of set V must be explicitly specified and the action of the operations $\oplus$ and $\odot$ must be stated. Hence we can check properties 1 and 2 and properties a through h directly. If any of these properties is not satisfied, then set V with operations $\oplus$ and $\odot$ is not a vector space.
EXAMPLES of REAL Vector Spaces

- $\mathbb{R}^n$ ==> n-vectors (columns or rows)
- All the $m \times n$ matrices with real entries. $\mathbb{R}^{m \times n}$
- All $n \times n$ diagonal matrices.
- All symmetric matrices.
- All $2 \times 2$ matrices of the following form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ where $a$, $b$, and $c$ are any real numbers.
- $\text{ns}(A)$ ==> the null space of any matrix.
- $\text{row}(A)$ ==> the row space of any matrix.
- $\text{col}(A)$ ==> the column space of any matrix.
- The eigenspace associated with an eigenvalue of $A$.
- All polynomials of degree $k$ or less. $\mathbb{P}_k$
- All polynomials. $\mathbb{P}$
- All functions continuous of $[a,b]$. $\mathbb{C}[a,b]$
- All functions continuous over all real numbers. $\mathbb{C}(\mathbb{R})$


Vector Space Properties

Let \((V, \oplus, \otimes)\) be a real (or complex) vector space, with vectors \(u, v,\) and \(w\) in \(V\), scalars \(c\) and \(k\), and identity vector \(z\).

i) \(0 \otimes u = z\) (zero times any vector is the identity vector)

ii) \(k \otimes z = z\) (any scalar multiple of the identity vector is the identity vector)

iii) If \(k \otimes u = z\), then either \(k = 0\) or \(u = z\).

iv) \((-1) \otimes u = -u\) (negative 1 times a vector is the negative of the vector)

v) \(-(-u) = u\) (the negative of vector \(-u\) is \(u\))

vi) If \(u \oplus v = u \oplus w\), then \(v = w\).

vii) If \(u \neq z\) and \(k \otimes u = c \otimes u\), then \(k = c\).
**Strategy:**

When dealing with a set $V$ and operations that are not familiar, checking the **ten vector properties** necessary for a vector space can be quite laborious.

It is recommended that we **first check properties 1 and 2**, the closures, **and then property c**, the existence of an identity or zero vector.

If these are satisfied then check the remaining seven properties. If any property fails to hold, then we can immediately stop and state that $(V, \oplus, \otimes)$ is **not** a vector space.

---

**Terminology**

<table>
<thead>
<tr>
<th>Vector space.</th>
<th>Closed with respect to $\oplus$ and $\otimes$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vectors.</td>
<td>Identity or zero vector.</td>
</tr>
<tr>
<td>$P_n; P.$</td>
<td>$C(a,b); C[a,b]; C(-\infty, \infty)$.</td>
</tr>
<tr>
<td>The zero vector space.</td>
<td></td>
</tr>
</tbody>
</table>

**Conventions:**

- Vector addition $u \oplus v$ will be denoted $u + v$.
- Scalar multiplication $k \otimes u$ will be denoted $ku$.
- The context of the example or discussion will tell us the particular definitions of vector addition and scalar multiplication.
- The identity vector $z$ will be called the **zero vector**.\(^1\)
  - In $\mathbb{R}^n$, $\mathbb{C}^n$, $R_{m\times n}$, and $C_{m\times n}$, we will denote the zero vector as $0$.
- The subspace $W = \{z\}$ will be called the **zero subspace**.

---

\(^1\) The identity vector or zero vector need not contain any zero entries or be zero. Its content depends upon the definitions of vector addition and scalar multiplication.
Example. Let $V$ be the set of all $2 \times 1$ real matrices with both entries positive. For vectors $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ in $V$, we define vector addition as

$$v \oplus w = \begin{bmatrix} v_1 w_1 \\ v_2 w_2 \end{bmatrix}$$

and scalar multiplication by a real scalar $k$ is defined as

$$k \otimes v = \begin{bmatrix} v_1^k \\ v_2^k \end{bmatrix}.$$

We claim $(V, \oplus, \otimes)$ is a real vector space. We verify properties 1, c, and e for a vector space and leave the remaining properties as exercises.
We have a bunch of familiar ideas associated with abstract vector spaces.

**Terminology**

<table>
<thead>
<tr>
<th>Subspace; Subspace Criterion.</th>
<th>Trivial subspaces.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subspaces connected with linear systems of equations.</td>
<td>Linear combination.</td>
</tr>
<tr>
<td>Span of a set.</td>
<td>Determining if the span of a set is the whole space.</td>
</tr>
</tbody>
</table>

**Subspace Criterion**

Let \((V, \oplus, \otimes)\) be a real (or complex) vector space. A subset \(W\) of \(V\) is a subspace if and only if

i) \(W\) is closed with respect to vector addition;
   that is, for \(r\) and \(s\) in \(W\), \(r \oplus s\) is in \(W\).

and

ii) \(W\) is closed with respect to scalar multiplication;
   that is, for \(r\) in \(W\) and \(k\) any scalar, \(k \otimes r\) is in \(W\).

**MORE FAMILIAR THINGS!**

**Terminolgy**

<table>
<thead>
<tr>
<th>Linearly dependent/independent sets.</th>
<th>Basis.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural basis or standard basis.</td>
<td>Dimension of a vector space.</td>
</tr>
</tbody>
</table>
PROBLEMS

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