Section 4.2 Computing Eigen Information for Small Matrices

- In Section 4.1 we used an algebraic approach to compute the eigenvalues and eigenvectors of a $2 \times 2$ matrix.

- In this section we use the eigen equation $A p = \lambda p$ together with determinants and the solution of homogeneous linear systems of equations to find eigenpairs.

- This approach is valid for $n \times n$ matrices of any size. However, some of the computational steps are quite tedious for $n > 4$ except for matrices with special structures.
The eigen equation can be rearranged as follows:

\[ A p = \lambda p \iff A p = \lambda I_n p \iff A p - \lambda I_n p = 0 \iff (A - \lambda I_n) p = 0 \]

(1)

- The matrix equation \((A - \lambda I_n) p = 0\) is a homogeneous linear system with coefficient matrix \(A - \lambda I_n\).

- Since an eigenvector \(p\) cannot be the zero vector, this means we seek a nontrivial solution to the linear system \((A - \lambda I_n) p = 0\).

- Thus \(\text{ns}(A - \lambda I_n) \neq 0\) or equivalently \(\text{rref}(A - \lambda I_n)\) must contain a zero row.

- It follows that matrix \(A - \lambda I_n\) must be singular, so from Chapter 2,

\[ \det(A - \lambda I_n) = 0. \]

(2)

- Equation (2) is called the characteristic equation of matrix \(A\) and solving it for \(\lambda\) gives us the eigenvalues of \(A\).

- Because the determinant is a linear combination of particular products of entries of the matrix, the characteristic equation is really a polynomial equation of degree \(n\). We call

\[ c(\lambda) = \det(A - \lambda I_n) \]

(3)

the characteristic polynomial of matrix \(A\).

- The eigenvalues are the solutions of (2) or equivalently the roots of the characteristic polynomial (3).

- Once we have the \(n\) eigenvalues of \(A\), \(\lambda_1, \lambda_2, \ldots, \lambda_n\), the corresponding eigenvectors are nontrivial solutions of the homogeneous linear systems
(A - \lambda I_n)p = 0 \quad \text{for } i = 1, 2, \ldots, n.

(4)

We summarize the computational approach for determining eigenpairs \((\lambda, p)\) as a two-step procedure:

**Step I.** To find the eigenvalues of \(A\) compute the roots of the characteristic equation \(\det(A - \lambda I_n) = 0\).

**Step II.** To find an eigenvector corresponding to an eigenvalue \(\mu\), compute a nontrivial solution to the homogeneous linear system \((A - \mu I_n)p = 0\).

**Example 1.** Let \(A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\). To find eigenpairs of \(A\) we follow the two step procedure given above.

**Example 2.** Let \(A = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}\). To find its eigenpairs we follow our two step procedure.
The eigenvalues are the roots of the characteristic polynomial, thus it is possible that a numerical value can be a root more than once. For example: \( \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda^2 - 6\lambda + 9) = \lambda(\lambda - 3)^2 \) has roots 0, 3, 3. We say 3 is a **multiple** or **repeated root**. The number of times the root is repeated is called its **multiplicity**. Thus it follows that a matrix can have repeated eigenvalues.

**Example 3.** Let \( A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \). To find eigenpairs of \( A \) we follow the two step procedure given above.
As we saw for Markov processes knowing a particular eigenvalue and possibly an associated eigenvector can provide information about the behavior of the model. In the case of a Markov process we wanted to determine a stable state which corresponds to an eigenvector associated with eigenvalue $\lambda = 1$. A natural question is,

'Does the transition matrix of a Markov process always have eigenvalue 1?'

To show that the answer is yes, we need another property of eigenvalues. In Section 4.1 we developed Properties I-IV, so we continue the list here with Property V.

Property V. $A$ and $A^T$ have the same eigenvalues.

We see this by showing that $A$ and $A^T$ have the same characteristic polynomial.

A transition matrix $A$ of a Markov process has entries $0 \leq a_{ij} \leq 1$ with the sum of the entries in each column equal to 1. Thus the sum of the entries in each row of $A^T$ is 1 and we have

$$A^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}.$$ 

Hence 1 is an eigenvalue of $A^T$ and also of transition matrix $A$. 
The characteristic equation is derived from a determinant; 
\[ c(\lambda) = \det(A - \lambda I_n) = 0. \] Thus matrices for which the determinant calculation is especially simple may yield their eigenvalues with little work. We have the following property.

**Property VI.** If \( A \) is diagonal, upper triangular, or lower triangular, then its eigenvalues are its diagonal entries.
Symmetric matrices arise frequently in applications. The next two properties provide information about their eigenvalues and eigenvectors.

Property VII. The eigenvalues of a symmetric matrix are real.

Property VIII. The eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.
For a square matrix $A$ we have the following important ideas:

Is $A$ nonsingular or singular? {See Chapter 2.}
Is $\det(A)$ nonzero or zero? {See Chapter 3.}

Next we connect these to properties of the eigenvalues.

**Property IX.** $|\det(A)|$ is the product of the absolute values of the eigenvalues of $A$.

We see this as follows:
If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$, then they are the roots of its characteristic polynomial. Hence
$$c(\lambda) = \det(A - \lambda I_n) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$
Evaluating this polynomial at $\lambda = 0$ gives
$$\det(A) = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$$
and so $|\det(A)| = |\lambda_1| |\lambda_2| \cdots |\lambda_n|$.

**Property X.** $A$ is nonsingular if and only if 0 is not an eigenvalue of $A$, or equivalently, $A$ is singular if and only if 0 is an eigenvalue of $A$.

We see this as follows:
$A$ is nonsingular if and only if $\det(A) \neq 0$.
Since $|\det(A)| = |\lambda_1| |\lambda_2| \cdots |\lambda_n| \neq 0$, zero cannot be an eigenvalue.
Terminology

<table>
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<th>Characteristic equation and polynomial.</th>
<th>Repeated eigenvalues.</th>
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<td>Defective matrix.</td>
<td>Power method.</td>
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From a computational point of view this section focussed on determining eigenvalues and eigenvectors using the algebraic tools of a determinant and the solution of a homogeneous linear systems. However, Properties V – X set the stage for further developments that utilize eigenvalues and eigenvectors in upcoming sections. It is these properties that let us extract pertinent information about a matrix based on its eigenpairs.

- What is the characteristic equation of a matrix? How many unknowns are there in the characteristic equation? Describe them.
- Describe the two-step procedure we developed for determining eigenpairs.
- Why is the two-step procedure not recommended for large matrices?
- What do we mean by a repeated eigenvalue?
- If a $3 \times 3$ matrix has eigenvalues $-2, 6, \text{and } 6$, how many linearly independent eigenvectors can it have? If there is more than one possible answer, explain each case.
- What is a defective matrix?

It is important to be able to use the special structure of certain matrices so that we can infer information regarding their eigenvalues and vectors without actually performing computations. Use this notion to respond to the following questions and statements.

- How are the eigenvalues of a matrix and its transpose related?
- How can we identify the eigenvalues of a triangular matrix? Of a diagonal matrix?
- What property do all the eigenvalues of a symmetric matrix share?
• If $(\lambda, \mathbf{p})$ and $(\mu, \mathbf{q})$ are eigenpairs of a symmetric matrix with $\lambda \neq \mu$, then how are vectors $\mathbf{p}$ and $\mathbf{q}$ related?
• How are the eigenvalues of a matrix related to its determinant?
• If we know all the eigenvalues of a matrix, how can we determine whether it is singular?
• What does the power method approximate?
• Describe the power method.