Chapter 4 Eigen Information

• Recall that a matrix transformation is, a function f from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) determined by an \( m \times n \) matrix \( A \) so that \( f(p) = Ap \), for \( p \) in \( \mathbb{R}^m \).

• Here we only consider the case where \( m = n \); that is, square matrices. We begin with \( n = 2 \) or \( 3 \) so that we can generate a geometric model that will aid in visualizing the concepts to be developed.

• Matrix transformations do not preserve length, that is \( ||Ap|| \) need not be the same as \( ||p|| \).

• Since a vector \( p \) in \( \mathbb{R}^n \) has two basic properties, length and direction; we investigate whether \( p \) and \( Ap \) are ever parallel for a given matrix \( A \). (This is a geometric point-of-view.)

• Note that \( A0 = 0 \), hence we need consider only nonzero vectors \( p \).

• We use geometry to begin with and soon switch to the more powerful corresponding ideas in algebra.
Section 4.1 Eigenvalues and Eigenvectors

- For a given $2 \times 2$ matrix $A$ we proceed pictorially to investigate parallel inputs and outputs of the matrix transformation $f(p) = Ap$. Choose an input vector $p$ with tip on the unit circle and then sketch the resulting output vector $Ap$.

- Use the MATLAB routine `matvec`.

- Next turn to the algebraic formulation of parallel which says that one of the vectors must be a scalar multiple of the other;

  $$Ap = (\text{scalar}) \cdot p.$$  

(1)
Example  Let \( A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). Let \( p = \begin{bmatrix} x \\ y \end{bmatrix} \).

Determine vectors \( p \) so that \( Ap \) is parallel to \( p \).

Example  Let \( A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \). Let \( p = \begin{bmatrix} x \\ y \end{bmatrix} \).

Determine vectors \( p \) so that \( Ap \) is parallel to \( p \).
**Definition**  Let $A$ be an $n \times n$ matrix. The scalar $\lambda$ is called an **eigenvalue** of matrix $A$ if there exists an $n \times 1$ vector $p$, $p \neq 0$, such that

$$Ap = \lambda p.$$  

(2)

Every nonzero vector $p$ satisfying (2) is called an **eigenvector of $A$ associated with eigenvalue $\lambda$**.

- We will use the term **eigenpair** to mean an eigenvalue and an associated eigenvector. This emphasizes that for a given matrix $A$ a solution of (2) requires both a scalar, an eigenvalue $\lambda$, and a nonzero vector, an eigenvector $p$.

- Equation (2) is commonly called the **eigen equation** and its solution is the goal of this chapter.

- In the eigen equation in (2) there is no restriction on the entries of matrix $A$, the type of scalar $\lambda$, or the entries of vector $p$, other than they cannot all be zero. **It is possible that an eigenpair of a matrix with real entries can involve complex values.** In such a case the geometric approach of parallel input and output fails to lead us to an appropriate solution.
In order to obtain solutions of the eigen problem we will use an algebraic approach rather than rely on geometry. With this in mind we can make some important observations about eigenvalues and eigenvectors directly from (2).

**Eigen Properties**

I. If \((\lambda, p)\) is an eigenpair of \(A\), then so is \((\lambda, kp)\) for any scalar \(k \neq 0\).

II. If \(p\) and \(q\) are eigenvectors corresponding to eigenvalue \(\lambda\) of \(A\), then so is \(p + q\) (assuming \(p + q \neq 0\)).

III. If \((\lambda, p)\) is an eigenpair of \(A\), then for any positive integer \(r\), \((\lambda^r, p)\) is an eigenpair of \(A^r\).
IV. If \((\lambda, p)\) and \((\mu, q)\) are eigenpairs of \(A\) with \(\lambda \neq \mu\), then \(p\) and \(q\) are linearly independent.

We see this as follows:

\((\lambda, p)\) is an eigenpair of \(A\) \(\Rightarrow\) \(Ap = \lambda p\).

\((\mu, q)\) is an eigenpair of \(A\) \(\Rightarrow\) \(Aq = \mu q\).

Let \(s\) and \(t\) be scalars so that \(tp + sq = 0\), or equivalently \(tp = -sq\).

Multiplying this linear combination by \(A\) we get

\[A(tp + sq) = t(Ap) + s(Aq) = t(\lambda p) + s(\mu q) = \lambda (tp) + \mu (sq) = 0.\]

Substituting for \(tp\) we get

\[\lambda(-sq) + \mu(sq) = 0 \iff s(\lambda - \mu)q = 0\]

Since \(q \neq 0\) and \(\lambda \neq \mu\), we have \(s = 0\), but then

\[tp = -0q = 0 \Rightarrow t = 0.\]

Hence the only way \(tp + sq = 0\) is when \(t = s = 0\), so \(p\) and \(q\) are linearly independent.

Property IV is often stated as:

*Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Note that Properties I-IV are valid for an eigenvalue \(\lambda = 0\).
In a Markov process the matrix transformation is used to form successive compositions as follows:

Let \( s_0 \) be an initial vector.
Then
\[
    s_1 = f(s_0) = As_0 \\
    s_2 = f(s_1) = f(f(s_0)) = As_1 = A^2s_0 \\
    \vdots
\]

(3)
\[
    s_{n+1} = f(s_n) = As_n = A^ns_0 \\
    \vdots
\]

If the sequence of vectors \( s_0, s_1, s_2, \ldots \) converges to a limiting vector \( p \), then we call \( p \) the steady state of the Markov process. (See Example 7 in Section 1.3.) If the process has a steady state \( p \), then

\[
    f(p) = p \iff Ap = p.
\]

That is, the steady state is an eigenvector corresponding to an eigenvalue \( \lambda = 1 \) of the transition matrix \( A \).

Example Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \) be the transition matrix of the Markov process. Find its steady state using eigen computations.
### Terminology

<table>
<thead>
<tr>
<th>Eigenvalue.</th>
<th>Eigenvector.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenpair.</td>
<td>Eigen equation.</td>
</tr>
<tr>
<td>Distinct eigenvalues.</td>
<td>Dominant eigenvalue.</td>
</tr>
<tr>
<td>Eigenspace.</td>
<td>Invariant subspace.</td>
</tr>
</tbody>
</table>

- Explain how the notion of parallel vectors is related to eigenvectors.
- If \((\lambda, p)\) is an eigenpair of matrix \(A\), then state a (matrix) algebraic relation that must be true.
- Explain why we require that an eigenvector not be the zero vector.
- Explain why we can claim that linear combinations of eigenvectors for a matrix \(A\) are guaranteed to generate another eigenvector of \(A\). (Are there any (technical) restrictions to this claim?)
- Let \(A\) be a \(2 \times 2\) matrix with a pair of distinct eigenvalues. Explain why we can say that the corresponding eigenvectors form a basis for \(\mathbb{R}^2\).
- Let \(A\) be the transition matrix of a Markov chain. How are eigenvalues and eigenvectors related to the long-term behavior of the process?
- Let \(A\) be a Leslie matrix. How are eigenvalues and eigenvectors related to the long-term behavior of the populations?
- What is an eigenspace?