Section 2.4 Nonsingular Linear Systems

Here we study square linear systems and properties of their coefficient matrices as they relate to the solution set of the linear system.

Let $A$ be $n \times n$. Then we know from previous work that

$$Ax = b$$

has a unique solution for any $n \times 1$ right side $b$ if and only if $\text{rref}(A) = I_n$, the $n \times n$ identity matrix.

- It follows that if $\text{rref}(A) = I_n$, then $\text{rref}([A | b]) = [I_n | c]$ and $x = c$ is the unique solution of $Ax = b$.

- In the case where $b = 0$, that is, a homogeneous linear system, $x = c = 0$ which implies the linear system has only the trivial solution.

- Coefficient matrices of square linear systems with unique solutions are important in their own right and we make the following definition.

Definition An $n \times n$ matrix $A$ is called nonsingular or invertible provided $\text{rref}(A) = I_n$.

Thus we say that a nonsingular linear system has a unique solution, meaning that the coefficient matrix of the linear system is square and its RREF is an identity matrix.

Next we develop an important property of nonsingular matrices. We will use linear combinations, linear systems, RREF, and properties of partitioned matrices. Recall that a matrix product $PQ$ can be viewed as $P$ times $Q$ partitioned into its columns, that is, $Q = [q_1 \ q_2 \ \ldots \ q_n]$, so

$$PQ = P[q_1 \ q_2 \ \ldots \ q_n] = [Pq_1 \ Pq_2 \ \ldots \ Pq_n] .$$

Thus each column of the product is really a linear combination of the columns of matrix $P$. 


If $A$ is an $n \times n$ nonsingular matrix, then for any $n \times 1$ matrix $b$, the linear system $Ax = b$ has a unique solution. Partition $I_n$ into its columns which we denote as $e_j$, $j = 1, 2, ..., n$; $I_n = [e_1 \ e_2 \ ... \ e_n]$. Then $Ax = e_j$ has a unique solution for each $j$. Denote the solution as $c_j$. Hence we have

$$Ac_1 = e_1, \ Ac_2 = e_2, \ ... \ , Ac_n = e_n.$$  \hspace{1cm} (1)

Writing (1) in matrix form, we have

$$[Ac_1 \ Ac_2 \ ... \ Ac_n] = [e_1 \ e_2 \ ... \ e_n] = I_n.$$

From our observation on partitioned matrices above, this equation can be written in the form

$$A[c_1 \ c_2 \ ... \ c_n] = I_n.$$

Letting $C = [c_1 \ c_2 \ ... \ c_n]$, we have just shown the following result.

**Statement I.** If $A$ is nonsingular then there exists a unique matrix $C$ such that $AC = I_n$.

Note that vector $c_j = \text{col}_j(C)$ is the unique solution of $Ax = e_j$, hence it follows that matrix $C$ is unique.

It is also true that

**Statement II.** If $A$ is nonsingular then there exists a unique matrix $D$ such that $DA = I_n$.

Next we show that $C = D$ as follows:

$$C = I_nC = (DA)C = D(CA) = DI_n = D.$$  \hspace{1cm} (2)

Thus combining Statements I and II we have

**Statement III.** If $A$ is nonsingular then there exists a unique matrix $C$ such that $AC = CA = I_n$. 
Statement III motivates the following definition.

**Definition** For an \( n \times n \) nonsingular matrix \( A \), the unique matrix \( C \) such that \( AC = CA = I_n \) is called the *inverse* of \( A \). We denote the inverse of \( A \) by the symbol \( A^{-1} \).

Now you see why nonsingular matrices are also called invertible matrices; they have an inverse. **Warning:** \( A^{-1} \) is not to be interpreted as a reciprocal\(^1\); it is just the notation to denote the inverse of matrix \( A \).

To complete this characterization of nonsingular matrices we also need to verify that if \( A \) has an inverse then \( \text{rref}(A) = I_n \). We do this at the end of this section.

To summarize things we present the following set of equivalent statements:

1. \( A \) is nonsingular.
2. \( \text{rref}(A) = I_n \).
3. \( A \) has an inverse.

We have tied together the unique solution of a square linear system, \( \text{RREF} \), and the matrix multiplication property \( AA^{-1} = A^{-1}A = I_n \). This leads us to the following observation.

A nonsingular linear system \( Ax = b \) has the unique solution \( x = A^{-1}b \).

This follows from the equivalent statements given next:

\[
Ax = b \text{ or } A^{-1}A x = A^{-1}b \text{ or } I_n x = A^{-1}b \text{ or } x = A^{-1}b
\]

The relations in (3) say that we have a formula for the unique solution of a nonsingular linear system and it can be computed in two steps:

i) Find \( A^{-1} \).

ii) Compute the matrix-vector product \( A^{-1}x \).\(^2\)

\(^1\) In the real or complex numbers, the term reciprocal means inverse.
The final result in (3) is advantageous for algebraic manipulation of expressions and theoretical purposes, but the solution of $Ax = b$ is more efficiently computed using the fact that $rref([A | b]) = [ I_n | c]$ implies that the unique solution is $x = c$.

Next we show how to compute the inverse of a matrix.

The procedure we develop simultaneously determines if a square matrix has an inverse, and if it does, computes it. From (1) we have that the columns of the inverse of $A$ are the solutions of the linear systems

$$Ax = \text{col}_j(A) = e_j, \ j=1,2, \ldots, n.$$ 

The corresponding set of augmented matrices can be combined into the following partitioned matrix

$$[A | e_1 | e_2 | \ldots | e_n] = [A | I_n].$$ 

From our previous arguments we have the following statement:

**Statement IV.** For $n \times n$ matrix, $rref([A | I_n]) = [ I_n | C]$ if and only if $A$ is nonsingular and $C = A^{-1}$.

It follows that if $rref([A | I_n]) \neq [ I_n | C]$, then $A$ has no inverse. We call a matrix with no inverse a **singular** matrix. Recall that our focus here is on square matrices. The terms singular and nonsingular do not apply to matrices which are not square.

**Example 2.** For each of the following matrices determine if it is nonsingular, and if it is, find its inverse.

a) $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 9 & 10 \\ 3 & 13 & 16 \end{bmatrix}$.
Example 3. Let \( \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be nonsingular. By carefully using row operations on \([\mathbf{A} | \mathbf{I}_2]\) we can develop an expression for \(\mathbf{A}^{-1}\) as follows. To start, we know that \(\text{rref}(\mathbf{A}) = \mathbf{I}_2\); hence one of the entries in column 1 of \(\mathbf{A}\) is not zero. Assume that \(a \neq 0\); if \(a = 0\), then we can interchange rows 1 and 2 below and proceed. Use the entry \(a\) as the first pivot and perform the following sequence of row operations:

\[
\begin{bmatrix}
 a & b & 1 & 0 \\
 c & d & 0 & 1 \\
\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}
 1 & b/a & 1/a & 0 \\
 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \\
\end{bmatrix}
\]

\[
\text{R}_1 \quad \Rightarrow \quad \begin{bmatrix}
 1 & b/a & 1/a & 0 \\
 0 & a & c/a & 1 \\
\end{bmatrix}
\]

Since \(\text{rref}(\mathbf{A}) = \mathbf{I}_2\) entry \(d - \frac{bc}{a} = \frac{ad - bc}{a} \neq 0\) (Why must this be true?) so we can use it as our second pivot to continue the reduction to RREF.

\[
\begin{bmatrix}
 1 & b/a & 1/a & 0 \\
 0 & a & c/a & 1 \\
\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}
 1 & 0 & \frac{1}{a} & 0 \\
 0 & 1 & -\frac{c}{ad - bc} & \frac{a}{ad - bc} \\
\end{bmatrix}
\]

Next we eliminate above the leading 1 in the second row:

\[
\begin{bmatrix}
 1 & b/a & 1/a & 0 \\
 0 & 1 & -\frac{c}{ad - bc} & \frac{a}{ad - bc} \\
\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}
 1 & 0 & \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\
 0 & 1 & \frac{-c}{ad - bc} & \frac{a}{ad - bc} \\
\end{bmatrix}
\]

Thus

\[
\mathbf{A}^{-1} = \begin{bmatrix}
 \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\
 \frac{-c}{ad - bc} & \frac{a}{ad - bc} \\
\end{bmatrix} = \begin{bmatrix}
 1 & \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\
 0 & -\frac{c}{ad - bc} & \frac{a}{ad - bc} \\
\end{bmatrix}.
\]
To use this formula we note that we compute the reciprocal of \((ad - bc)\), interchange the diagonal entries, and change the sign of the off-diagonal entries. Following this prescription for \(A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\) we have

\[
\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}^{-1} = \frac{1}{(3)(0) - (2)(-1)} \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.
\]

We also note that if \(ad - bc = 0\), then the \(2 \times 2\) matrix has no inverse; that is, it is singular. This computation, \(ad - bc\), is often used as quick method to test if a \(2 \times 2\) matrix is nonsingular or not. We will see this in more detail in Chapter 3.

For \(3 \times 3\) matrices a development similar to that in Example 3 can be made. The result is a good deal more complicated.

Let \(w = iae - afh - idb + dch + gbf - gce\). Then

\[
\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{w} \begin{bmatrix} ie - fh & -(ib - ch) & bf - ce \\ -id + fg & ia - cg & -(af - cd) \\ -(dh + eg) & -(ah - bg) & ae - bd \end{bmatrix}.
\]

If the value of \(w = 0\), then the \(3 \times 3\) matrix is singular. Computationally it is more realistic to compute \(rref([A | I])\) and extract \(A^{-1}\) as we did in Example 2a.
The computation of the inverse of a nonsingular matrix can be viewed as an operation on the matrix and need not be linked to the solution of a linear system. The inverse operation has algebraic properties in the same way that the transpose of a matrix had properties. *We adopt the convention that if we write the symbol \( A^{-1} \), then it is understood that \( A \) is nonsingular.* Using this convention we next list properties of the inverse.

**Algebraic Properties of the Inverse**

i) \((AB)^{-1} = B^{-1}A^{-1}\) The inverse of a product is the product of the inverses in reverse order.

ii) \((A^T)^{-1} = (A^{-1})^T\) When \( A \) is nonsingular so is \( A^T \) and the inverse of \( A^T \) is the transpose of \( A^{-1} \).

iii) \((A^{-1})^{-1} = A\) When \( A \) is nonsingular so is \( A^{-1} \) and its inverse is \( A \).

iv) \((kA)^{-1} = (1/k) \ A^{-1} \quad k \neq 0\) When \( A \) is nonsingular, so is any nonzero scalar multiple of it.

v) \((A_1A_2 \ldots A_r)^{-1} = A_r^{-1} \ldots A_2^{-1} A_1^{-1}\) The inverse of a product extends to any number of factors.

The verification of the preceding properties uses the fact that a matrix \( C \) is the inverse of an \( n \times n \) matrix \( D \) provided \( CD = I_n \), or \( DC = I_n \). This is the alternative to the definition of nonsingular matrices given in Statement III earlier. We illustrate the use of this equivalent characterization of nonsingular matrices in Example 4 where we verify i) and leave the verification of the remaining properties as exercises.

**Example 4.** Show that \((AB)^{-1} = B^{-1}A^{-1}\).
A common thread that has linked our topics in this chapter is linear combinations. It is unfortunate that the operation of computing the inverse does not follow one of the principles of linear combinations. Namely $(A + B)^{-1}$ is not guaranteed to be $A^{-1} + B^{-1}$. We illustrate this failure and more in Example 5.

**Example 5.** a) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 6 \\ 0 & 1 \end{bmatrix}$. Both $A$ and $B$ are nonsingular (verify), but $A + B = \begin{bmatrix} 0 & 8 \\ 0 & 2 \end{bmatrix}$ is not since its RREF is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq I_2$. So $A + B$ is singular.

b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Both $A$ and $B$ are singular. (Verify.) However, $A + B = I_2$ is nonsingular. (Verify.)

Hence nonsingular matrices are not closed under addition, nor are singular matrices.

As we have seen, the construction of the inverse of a matrix can require quite a few arithmetic steps which are carried out using row operations. However, there are several types of matrices for which it is easy to determine by observation if they are nonsingular. We present several such results next.

**Statement V.** A diagonal matrix, a lower triangular matrix, and an upper triangular matrix are nonsingular if and only if all the diagonal entries are different from zero.
For a nonsingular diagonal matrix $D$ we can explicitly construct its inverse, $D^{-1}$, as we show in Example 6.

Example 6. Let $D$ be an $n \times n$ nonsingular diagonal matrix. By Statement V we have $D = 
\begin{bmatrix}
  d_{11} & 0 & \cdots & 0 & 0 \\
  0 & d_{22} & \cdots & 0 & 0 \\
  0 & 0 & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & 0 & \vdots \\
  0 & 0 & \cdots & 0 & d_{nn}
\end{bmatrix}$

where $d_{jj} \neq 0$, $j = 1, 2, \ldots, n$. To construct $D^{-1}$ we compute $\text{rref}([D \ | \ I_n])$. Since the only nonzero entries are the diagonal entries we use row operations to produce leading 1's. It follows that the $n$ row operations

$\left(\frac{1}{d_{11}}\right)R_1,
\left(\frac{1}{d_{22}}\right)R_2,
\ldots,
\left(\frac{1}{d_{nn}}\right)R_n$

applied to $[D \ | \ I_n]$ give

$\text{rref}([D \ | \ I_n]) = [I_n \ | \ D^{-1}] =
\begin{bmatrix}
  1 & 0 & \cdots & 0 & \frac{1}{d_{11}} & 0 & \cdots & 0 & 0 \\
  0 & 1 & \cdots & 0 & 0 & \frac{1}{d_{22}} & \cdots & 0 & 0 \\
  0 & 0 & \ddots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots & 0 \\
  0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \frac{1}{d_{nn}}
\end{bmatrix}$. 

Hence $D^{-1}$ is a diagonal matrix whose diagonal entries are the reciprocals of the diagonal entries of $D$. 

\[ \square \]
We next investigate the subspaces associated with a nonsingular matrix. The key to verifying each of the following statements is that the RREF of a nonsingular matrix is an identity matrix. Let \( A \) be an \( n \times n \) nonsingular matrix, then we have the following properties.

**Properties of a Nonsingular Matrix \( A \)**

- i) A basis for \( \text{row}(A) \) ( \( \text{col}(A) \) ) are the rows ( columns ) of \( I_n \).
- ii) \( \text{ns}(A) = 0 \).
- iii) The rows ( columns ) of \( A \) are linearly independent.
- iv) \( \dim(\text{row}(A)) = n \) and \( \dim(\text{col}(A)) = n \).
- v) \( \text{rank}(A) = n \).

It is often difficult to determine by inspection whether a square matrix \( A \) is singular or nonsingular. For us, at this time, the determination depends upon looking at the RREF. However, since the RREF of a nonsingular matrix must be an identity matrix sometimes an inspection of \( A \) can tell us that this is not possible, hence we can claim that \( A \) is singular. The following is a list of occurrences that immediately imply that \( A \) is singular.

**Ways to Recognize Singular Matrices**

- i) \( A \) has a zero row or a zero column.
- ii) \( A \) has a pair of rows (columns) that are multiples of one another.
- iii) As we row reduce \( A \), a zero row appears.
- iv) The rows (columns) of \( A \) are linearly dependent.
- v) \( \text{ns}(A) \) contains a nonzero vector.