Chapter 2. Linear Systems and their Solution

It is the purpose of this chapter to establish procedures for efficiently determining the solution of a system of linear equations.

We know that the data within a system of linear equations can be represented in matrix form. We will show how to use the matrix representation with several simple arithmetic procedures to reveal information about any solutions. (There may be no solutions.) We will develop a systematic way to represent and analyze the behavior of linear systems.

Section 2.1 Linear Systems of Equations
A linear system of m equations in n unknowns \( x_1, x_2, ..., x_n \) is expressed as

\[
\begin{align*}
\text{(1)} \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n &= b_j \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]
The coefficient matrix is
\[ A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & \cdots & a_{mn}
\end{bmatrix} \]

and the right side is \( \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \).

We have shown that this linear system can be written as the matrix equation
\[ Ax = b \]
where \( x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \).

The information contained in the linear system can be represented by the partitioned matrix \([A \mid b]\), which we call the augmented matrix of the linear system. The right side \( \mathbf{b} \) is called the augmented column of the partitioned matrix.
Relating the Linear System to Matrix Vector Products

We know that the matrix product $Ax$ is a linear combination of the columns of $A$; that is, $Ax = \sum_{i=1}^{n} x_i \text{col}_i(A)$. Hence when we are asked to solve the matrix equation $Ax = b$ we seek a vector $x$ so that

vector $b = \sum_{i=1}^{n} x_i \text{col}_i(A)$. This is the same as saying that linear system $Ax = b$ has a solution if and only if $b$ is in

$$\text{span}\{\text{col}_1(A), \text{col}_2(A), \ldots , \text{col}_n(A)\}.$$  

Definition  The linear system $Ax = b$ is called consistent provided $b$ is in $\text{span}\{\text{col}_1(A), \text{col}_2(A), \ldots , \text{col}_n(A)\}$, otherwise the linear system is called inconsistent.

Thus a consistent linear system has a solution while an inconsistent linear system has no solution, and conversely.
A major goal is to be able to identify whether a linear system $Ax = b$ is consistent or inconsistent and for those that are consistent to determine a solution vector $x$ so that the product $Ax$ yields $b$. Thus merely being presented with the matrix equation $Ax = b$ does not imply that the linear system is consistent.

**Definition**  A linear system $Ax = b$ is called **homogeneous** provided the right side $b = 0$. Every homogeneous linear system can be represented as a matrix equation of the form $Ax = 0$.

A linear system $Ax = b$ is called **nonhomogeneous** provided $b \neq 0$.

- Every homogeneous system $Ax = 0$ is consistent.  
  (Why?)
- A nonhomogeneous system $Ax = b$ may be consistent, but there is no guarantee.

**Terminology:**
- **Trivial solution**
- **Nontrivial solution**
- **Unique solution**
- **Multiple solutions**

**Solution set**
- Possible types of solution sets.
A basic strategy in determining the solution set of a linear system $Ax = b$ is to manipulate the information in the coefficient matrix $A$ and right side $b$ to obtain a simpler system without changing the solution set.

By simpler system we mean a linear system from which it is easier to determine the solution set.

Given the intimate connection of linear systems to linear combinations we suspect that the manipulations that preserve the solution set will involve linear combinations.

We next investigate three fundamental manipulations that will form the basis for our solution techniques.

(Discussion.)

We call these row operations and abbreviate them as

- $R_i \leftrightarrow R_j$, meaning interchange rows $i$ and $j$.
- $kR_i$, meaning multiply row $i$ by a nonzero scalar.
- $kR_i + R_j$, meaning linear combination $kR_i + R_j$ replaces row $j$.

**Definition**  A pair of matrices is said to be row equivalent provided one matrix can be obtained from the other by a finite sequence of row operations.
Some Linear Systems that are EASY to Solve.

Examples.

What do you do if the system is not one of the EASY kind?

**Strategy:** For $Ax = b$, the augmented matrix is $[A \ | \ b]$. Apply row operations to obtain row equivalent matrices until you have an upper triangular form for the coefficient matrix, then apply back substitution.

Examples.

We call this strategy (row) **reduction to (upper) triangular** form followed by back substitution or (row) **reduction** for short. In addition to this strategy we make the following observations:

> If at any step of the reduction to triangular form process there appears a row of the form $[0 \ 0 \ ... \ 0 \ | \ ≠ 0]$, then the linear system is inconsistent and we need proceed no further.

> We may continue the reduction process until we get a diagonal or close to diagonal coefficient matrix which often simplifies the back substitution process.
More Terminology:

Arbitrary constant or Free Variable

Pivot

Pivot Row

Degrees of Freedom.