Section 1.4  Geometry of Linear Combinations

Previously we showed how linear combinations could be used to build new vectors and matrices and how matrix products are constructed from linear combinations. In this section we explore a geometric representation of linear combinations of vectors, concentrating on 2-vectors and 3-vectors.
Let $R^n$ denote the set of all $n$-vectors with real entries. $R^n$ is called *n-space*. Whether we regard a vector $\mathbf{a}$ in $R^n$ as a row or column will be determined by the context in which we use the vector. Recall that we have shown that the following properties hold in $R^n$:

$R^n$ is closed under both addition of vectors and scalar multiplication.

For $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $R^n$ and scalars $r$ and $k$ we have

$$
\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad \{\text{Commutativity of Addition}\}
$$

$$
\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \quad \{\text{Associativity of Addition}\}
$$

$$
\begin{align*}
  k(\mathbf{a} + \mathbf{b}) &= ka + kb \\
  (k + r)\mathbf{a} &= ka + ra \\
  k(r\mathbf{a}) &= (kr)\mathbf{a}
\end{align*}
$$

These properties are the tools for manipulating linear combinations.
There are three other properties of addition and scalar multiplication in $\mathbb{R}^n$, which we list now:

- There is a unique vector $\mathbf{z}$, so that $\mathbf{a} + \mathbf{z} = \mathbf{a}$, for any vector $\mathbf{a}$. $\mathbf{z}$ is called the zero vector and $\mathbf{z} = [0 \ 0 \ ... \ 0]$ in $\mathbb{R}^n$. We use $\mathbf{0}$ to denote the zero vector in $\mathbb{R}^n$.

- For a vector $\mathbf{b}$, there exists a unique vector $\mathbf{x}$, so that $\mathbf{b} + \mathbf{x} = \mathbf{0}$, the zero vector. We call $\mathbf{x}$ the negative of $\mathbf{b}$ and denote it as $-\mathbf{b}$ or equivalently as $(-1)\mathbf{b}$.

- For any vector $\mathbf{b}$, $1\mathbf{b} = \mathbf{b}$. 
THREE REPRESENTATIONS

A member of $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{R}^n$ has three interpretations:

- As a **matrix**.
- As a **point**.
- As a **vector**; geometrically a directed line segment.

It is easy to "see" this in $\mathbb{R}^2$ and $\mathbb{R}^3$, but we have no way to draw pictures in $\mathbb{R}^n$, $n > 3$.

Such multiple interpretations give us flexibility regarding the information within the members of $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^n$. **We can choose the form best suited for a particular situation.**
Next we develop geometric interpretations of linear combinations in $\mathbb{R}^2$, which generalize on an intuitive basis to $\mathbb{R}^n$.

Let $\mathbf{a}$ be in $\mathbb{R}^2$. Then for a scalar $k$, $k\mathbf{a} = [ka_1 \ ka_2]$. Figure 4 gives us a pictorial representation of the vector $k\mathbf{a}$ depending upon the value of $k$.

From Figure 4 we see that scalar multiplication can change the length of a vector while maintaining the same direction when $k > 0$ or reversing the direction for $k < 0$. If $k \geq 1$ then $k\mathbf{a}$ is called a dilation (or a stretching) of $\mathbf{a}$, while for $0 < k < 1$ we have that $k\mathbf{a}$ is called a contraction (or a shrinking) of $\mathbf{a}$. 

![Figure 4](image-url)
**Length of a Vector**

To compute the length or magnitude of a vector \( \mathbf{a} \) we use its representation as a directed line segment starting at the origin and terminating at the point determined by \( \mathbf{a} \). Thus in \( \mathbb{R}^2 \) we compute the distance from \((0,0)\) to \((a_1, a_2)\):

\[
\text{distance from } (0,0) \text{ to } (a_1, a_2) = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2} = \sqrt{a_1^2 + a_2^2}
\]

(Note the ease with which we switched from matrix to vector to point.) The length of vector \( \mathbf{a} \) is denoted by \( \| \mathbf{a} \| \). Thus we have

\[
\| k \mathbf{a} \| = \sqrt{(ka_1 - 0)^2 + (ka_2 - 0)^2} = \sqrt{k^2a_1^2 + k^2a_2^2} = k\sqrt{a_1^2 + a_2^2}
\]

Hence scalar multiplication alters, or scales, the length of a vector by the absolute value of the scalar. The idea of length generalizes to \( \mathbb{R}^n \), where for \( \mathbf{x} \) in \( \mathbb{R}^n \),

\[
\| \mathbf{x} \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{j=1}^{n} x_j^2}
\]
Example  Let $\mathbf{a} = [2 \ 3]$, $\mathbf{b} = [-1 \ 0 \ 4]$, and $\mathbf{c} = [1 \ 2 \ -1 \ -4 \ 0]$.

a) Compute $5\mathbf{a}$ and $\|5\mathbf{a}\|$.

b) Compute $\|\mathbf{b}\|$.

c) Compute $\|3\mathbf{c}\|$.
Alternate way to express the length of a vector.

The expression for the length or magnitude of a vector \( \mathbf{x} \) in \( \mathbb{R}^n \) can be written in another way. From the definition of dot product in Section 1.2 we have that \( \mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2 \), so that

\[
\Vert \mathbf{x} \Vert = \sqrt{\mathbf{x} \cdot \mathbf{x}}
\]

This is a convenient connection between the algebraic notion of the dot product and the geometric visualization of members of \( \mathbb{R}^n \) as vectors. This expression for the length of a vector will be used frequently.
Parallel Vectors

We can give another interpretation of a scalar multiple of a member of $\mathbb{R}^n$. We see from Figure 4 that if $b = ka$, then geometrically the vectors $a$ and $b$ are parallel; that is, they point either in the same direction or in opposite directions. Hence $a$ and $b$ are parallel vectors provided there is a scalar $k$ so that $b = ka$; that is, they are scalar multiples of one another.
A Geometric Model for Sums and Differences of Vectors in $\mathbb{R}^2$

- Parallelogram Rule
- Head-to-Tail Construction
Angles between vectors in $\mathbb{R}^n$

We define the cosine of the angle $\theta$ between the vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ by the expression

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b} - 2\left(\mathbf{a} \cdot \mathbf{b}\right) + \left\| \mathbf{b} \right\|^2 - \left\| \mathbf{a} \right\|^2}{-2\left\| \mathbf{a} \right\| \left\| \mathbf{b} \right\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\left\| \mathbf{a} \right\| \left\| \mathbf{b} \right\|}.$$ 

Since $-1 \leq \cos(\theta) \leq 1$ we must be sure that the expression $\frac{\mathbf{a} \cdot \mathbf{b}}{\left\| \mathbf{a} \right\| \left\| \mathbf{b} \right\|}$ has a value in $[-1, 1]$ for $n > 3$. The key to showing this result is the Cauchy-Schwartz-Buniakovsky inequality, whose proof we omit at this time, which states that $|\mathbf{a} \cdot \mathbf{b}| \leq \left\| \mathbf{a} \right\| \left\| \mathbf{b} \right\|$ for any vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. With this result it follows that if neither $\mathbf{a}$ nor $\mathbf{b}$ is the zero vector, then

$$-1 \leq \frac{\mathbf{a} \cdot \mathbf{b}}{\left\| \mathbf{a} \right\| \left\| \mathbf{b} \right\|} \leq 1.$$ 

Examples:

Find the angle between vectors $\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$.

Find the angle between vectors $\mathbf{c} = [1, 2, 0, 3]$ and $\mathbf{d} = [-2, 1, 7, 0]$. 

Orthogonal Vectors

If $a$ and $b$ are perpendicular, that is, if the angle between them is $\frac{\pi}{2}$ radians, then the law of cosines implies that $0 = \cos \left( \frac{\pi}{2} \right) = \frac{a \cdot b}{\|a\| \|b\|}$ and thus it follows that $a \cdot b = 0$. (This assumes that neither $a$ nor $b$ is the zero vector, which has length 0.) In summary, we have that nonzero vectors $a$ and $b$ are perpendicular or orthogonal if and only if $a \cdot b = 0$. (Again notice the close connection between geometry and algebra.) We will use the term orthogonal from now on.
Application: Translations
Section 1.4
True/False Review Questions

1. The length of vector $\mathbf{x}$ in $\mathbb{R}^n$ is $\mathbf{x} \cdot \mathbf{x}$.

2. For $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$, $\|\mathbf{v}\|$ is the same as the distance from the origin $(0,0)$ to the point $(v_1,v_2)$.

3. In $\mathbb{R}^2$, the sum of the vectors $\mathbf{a}$ and $\mathbf{b}$ is, geometrically, a diagonal of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$.

4. If vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ satisfy $\mathbf{x} \cdot \mathbf{y} = 0$ then $\mathbf{x}$ is parallel to $\mathbf{y}$.

5. A unit vector in the same direction as $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

6. Two vectors are orthogonal if the angle between them is $\pi$ radians.

7. If vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ are such that $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) = 0$, then $\|\mathbf{a}\| = \|\mathbf{b}\|$.

8. Vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ are parallel provided one is a scalar multiple of the other.

9. $\|k \mathbf{a}\| = k \|\mathbf{a}\|$

10. $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$
**Terminology**

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\mathbb{R}^n$; n-space</td>
<td>Threefold interpretation of $\mathbf{a}$ in $\mathbb{R}^2$.</td>
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<tr>
<td>Dilation; contraction</td>
<td>The length of an $n$-vector.</td>
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<tr>
<td>Parallel vectors</td>
<td>Parallelogram rule.</td>
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<tr>
<td>Angle between vectors in $\mathbb{R}^n$</td>
<td>Orthogonal vectors.</td>
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<tr>
<td>Unit vector</td>
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Algebra and geometry are different sides of the same coin. The concepts in this section showed aspects of both sides of the coin. It will be convenient to switch from side-to-side in order to provide different perspectives on computations and concepts. Answer the following to help yourself see various interpretations of some of the concepts in this section.

- List the three interpretations of a vector $\mathbf{a}$ in $\mathbb{R}^2$. Then give an example of a situation in which each interpretation has been used.
- Describe the relationship between scalar multiples and length in a dilation; in a contraction; for parallel vectors.
- Explain in your own words the parallelogram rule for adding two vectors (geometrically) in $\mathbb{R}^2$.
- How is the difference of two vectors depicted in the parallelogram rule?
- How do we compute the angle between two $n$-vectors?
- What is the angle between parallel vectors?
- What is the angle between orthogonal vectors?
- Explain how to construct a unit vector from a nonzero vector $\mathbf{v}$ of $\mathbb{R}^n$. 