Section 1.2 Matrix Operations

Definition Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if $\text{ent}_{ij}(A) = \text{ent}_{ij}(B)$, or equivalently $a_{ij} = b_{ij}$, for $1 \leq i \leq m$, $1 \leq j \leq n$, that is, if corresponding entries are equal.

This sounds simple and it is! Yet it is the key to verifying a number of properties involving matrix operations, as we will see.
Two matrices are equal if

- Corresponding entries are equal
- Corresponding rows are equal
- Corresponding columns are equal

To show two matrix expressions are equal, we show that corresponding entries of the expressions are equal.

Breaking a matrix up into rows or columns is one way to group information contained in a matrix. There are other ways:

**Definition** A submatrix of $A = [a_{ij}]$ is any set of entries obtained by omitting some, but not all, of its rows and/or columns.

The rows of a matrix are particular submatrices.

The columns of a matrix are particular submatrices.
Example Let \( B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \). Each of the following is a submatrix of \( B \). (Determine which rows and columns were omitted to obtain the submatrix.)

\[
Q_1 = \begin{bmatrix} 1 \\ 5 \\ 11 \\ 15 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 3 & 4 \\ 7 & 8 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 9 & 10 \\ 13 & 14 \end{bmatrix},
\]

\[
Q_4 = \begin{bmatrix} 11 & 12 \\ 15 & 16 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 1 & 3 \\ 9 & 11 \\ 13 & 15 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 4 \\ 13 & 16 \end{bmatrix}
\]
A matrix can be partitioned into submatrices or blocks by drawing horizontal lines between rows and vertical lines between columns. Each of the following is a partition of matrix $B$ from Example 1 and we see that a partitioning can be carried out in many different ways.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{bmatrix},
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{bmatrix},
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{bmatrix}
\]
Another type of matrix:

\[
A = \begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
3 & 4 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 6 & 7 \\
0 & 0 & 0 & 8 & 9 \\
\end{bmatrix}
\]

is called block diagonal since we can partition it into blocks which lie on the diagonal and all non-diagonal blocks contain only zeros.

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
3 & 4 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 6 & 7 \\
0 & 0 & 0 & 8 & 9 \\
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
3 & 4 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 6 & 7 \\
0 & 0 & 0 & 8 & 9 \\
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
3 & 4 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 6 & 7 \\
0 & 0 & 0 & 8 & 9 \\
\end{bmatrix}
\]
Every linear system of equations can be represented by a matrix which is partitioned into submatrices in a special way.

We can represent the linear system of equations

\[
\begin{align*}
    a + b + c &= -5 \\
    a - b + c &= 1 \\
    4a + 2b + c &= 7
\end{align*}
\]

by the matrix

\[
\begin{bmatrix}
    1 & 1 & 1 & -5 \\
    1 & -1 & 1 & 1 \\
    4 & 2 & 1 & 7
\end{bmatrix}
\]

Later we partition it into submatrices as follows:

\[
\begin{bmatrix}
    1 & 1 & 1 & -5 \\
    1 & -1 & 1 & 1 \\
    4 & 2 & 1 & 7
\end{bmatrix}
\]

where submatrix \( A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \) is called the coefficient matrix of the linear system and submatrix \( b = \begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix} \) is called the right side of the linear system. This particular partitioning is called an augmented matrix.
OPERATIONS on Matrices

Let $A$ and $B$ be matrices of the same size, then

- Their sum is $A + B$ and is formed by adding corresponding entries. $\text{ent}_{ij}(A + B) = \text{ent}_{ij}(A) + \text{ent}_{ij}(B)$

- Their difference is $A - B$ and is formed by subtracting corresponding entries. $\text{ent}_{ij}(A - B) = \text{ent}_{ij}(A) - \text{ent}_{ij}(B)$

- A scalar multiple of matrix $A$ is a number $k$ multiplied times each entry of the matrix; denoted $kA$. $\text{ent}_{ij}(kA) = k \cdot \text{ent}_{ij}(A)$

- The scalar multiple $(-1)A$ is often written $-A$, and is called the negative of $A$. Thus the difference $A - B = A + (-1)B$. The difference $A - B$ is also referred to as subtracting $B$ from $A$. 
LINEAR COMBINATIONS

We can combine scalar multiples of matrices with addition and subtraction into expressions of the form \(3A - 5B + C\), where it is assumed that matrices \(A\), \(B\), and \(C\) are the same size. The scalars multiplying the matrices are called coefficients or weights.

Definition  A linear combination of two or more quantities is a sum of scalar multiples of the quantities. The linear combination in which each scalar is zero is called the trivial linear combination.

Linear combinations are the most useful construction process in linear algebra. Much of what we do in this course utilizes linear combinations.
Example Let \( \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \) and \( \mathbf{d} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}. \) Then \( 2\mathbf{b} - \mathbf{c} + 3\mathbf{d} \) is a linear combination of the 2-vectors \( \mathbf{b}, \mathbf{c}, \) and \( \mathbf{d} \) with coefficients 2, -1, and 3 respectively. We have

\[
2\mathbf{b} - \mathbf{c} + 3\mathbf{d} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 16 \\ 1 \end{bmatrix}.
\]
Example  Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Then

$$c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4$$

is a linear combination of the $2 \times 2$ matrices $A_i$, $i = 1,\ldots,4$ with coefficients $c_1$, $c_2$, $c_3$, and $c_4$ respectively.
Example Polynomial $2x^3 - 7x^2 + 8x + 4$ is a linear combination of functions $x^3$, $x^2$, $x$, and 1 with coefficients 2, -7, 8, and 4 respectively.
Example Function $4 + 3\sin(x) + \sin(2x) - 2\cos(x) + 0.5\cos(2x)$ is a linear combination of the functions 1, $\sin(x)$, $\sin(2x)$, $\cos(x)$, and $\cos(2x)$ with weights 4, 3, 1, -2, and 0.5 respectively.
Span

Linear combinations play an important role in our analysis of the information contained in a matrix that represents a linear system of equations or other processes. The following two questions involving linear combinations are the focus of some future topics.

1. Given a quantity, can it be expressed as a linear combination of a particular set of quantities?

2. Given a particular set of quantities, what is the nature of the set of all possible linear combinations of these quantities?

We use the following terminology:

> The set of all possible linear combinations of a particular set $S$ of quantities is called the span of set $S$ and is denoted $\text{span}(S)$.

> If it is possible to find scalars so that a given quantity can be expressed as a linear combination of a particular set of quantities, we say that the given quantity is in the span of the set.
Example  Given the set of 2-vectors \( S = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix} \) and 2-vector \( b = \begin{bmatrix} 7 \\ 0 \end{bmatrix} \). Is \( b \) in span(\( S \))?

Example  Given the set of matrices

\[
S = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix}
\]

describe the set of all linear combinations of the members of \( S \); that is, describe span(\( S \)).
Properties of Matrix Addition.

Let $A$, $B$ and $C$ be matrices of the same size, then

(a) $A + B = B + A$ \hspace{1cm} \{Commutativity of Addition\} \quad \Leftarrow

(b) $A + (B + C) = (A + B) + C$ \hspace{1cm} \{Associativity of Addition\}

Properties of Linear Combinations.

Let $r$ and $s$ be scalars and $A$ and $B$ be matrices, then

(a) $r(sA) = (rs)A$

(b) $(r+s)A = rA + sA$ \quad \Leftarrow

(c) $r(A+B) = rA + rB$
Properties of the Transpose.

Let $r$ be a real scalar and $A$ and $B$ be matrices with real entries, then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$ \{The transpose of a sum is the sum of the individual transposes.\}

(c) $(rA)^T = r(A^T)$
Closure

Let $S$ be a set of quantities on which we have defined two operations; one called addition which operates on a pair of members of $S$ and a second called scalar multiplication which combines a scalar and a member of $S$.

We say that $S$ is **closed** if every linear combination of members of $S$ belongs to the set $S$. This can also be expressed by saying that the sum of any two members of $S$ belongs to $S$ (**closed under addition**) and that the scalar multiple of any member of $S$ belongs to the set $S$ (**closed under scalar multiplication**).

We sometimes refer to this as the **closure property** of a set and it implies closure with respect to both addition and scalar multiplication.
The key for determining whether a set S is closed or not, is to carefully determine the criteria for membership to the set S.

While this seems easy, it is at the heart of closure since we must add two members of S and ask is the result in S and take a scalar times a member of S and ask is the result also in S. If the answer to either question is no, then the set is not closed. This essentially means that we are not guaranteed that linear combinations of members of set S produce quantities that are also in S.

The best training in this important concept is to do lots of exercises, since each set S can be different in its criteria for membership.
Example  Let $S$ be the set of all $3 \times 1$ matrices of the form \[
\begin{bmatrix}
a \\
b \\
1
\end{bmatrix}
\]
where $a$ and $b$ are any real numbers.

What is the criteria for membership to set $S$? 
(State several members of $S$.)

Is set $S$ closed?
Example  Let T be the set of all $2 \times 2$ matrices of the form \[
\begin{bmatrix}
  a & 0 \\
  b & c
\end{bmatrix}
\]
where $c = 2a$ and $a$ and $b$ are any real numbers.

What is the criteria for membership to set $S$? (State several members of $S$.)

Is set $S$ closed?
Example  Let $V$ be a set of two $3 \times 1$ matrices.
(You can choose them, but once you make your choices they are fixed.)

Denote the vectors in $V$ as $u$ and $w$.

Let $S = \text{span}(V)$. Determine if $S$ is a closed set.
Applications:

RGB Color Displays

Polynomials

Parametric Form of a Line in the Plane