Trigonometric Curves with Sines & Cosines + Envelopes

Terminology:

AMPLITUDE – the maximum height of the curve
For any periodic function, the amplitude is defined as |M – m|/2
where M is the maximum value and m is the minimum value,
provided they exist.

PERIOD – A function f with domain set S is periodic if there exists a positive real
number k so that f(t + k) = f(t) for all t in S. If a least such positive real
number k exists, it is called the period of f.

The period is the “time” required for one complete ‘oscillation” of the
function.

ANGULAR FREQUENCY – The number of “oscillations” that occur in length of $2\pi$.

If the curve is given by the equation $A\sin(\omega t)$ or $A\cos(\omega t)$, then

$$\text{Amplitude} = |A|, \quad \text{Period} = \frac{2\pi}{\omega}, \quad \text{Angular Frequency} = \omega$$

FREQUENCY of MOTION – The frequency (of motion) is $1/(2\pi/\omega) = \omega/(2\pi)$,
which gives the number of oscillations per unit time.

Amp = 1
Per = $2\pi$
$\omega = 1$

Amp = 1
Per = $\pi$
$\omega = 2$

Amp = 1
Per = $\pi/2$
$\omega = 4$
Consider a combination of sine and cosine as follows:

\[ f(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \]  \hspace{1cm} (1)

This expression can be written in a more convenient compact form \( f(t) = A \sin(\omega t + \phi) \) using the following procedure. Note that

\[ f(t) = A \sin(\omega t + \phi) = A \cos(\omega t) \sin(\phi) + A \sin(\omega t) \cos(\phi) \]

Next let \( K_1 = A \sin(\phi) \) and \( K_2 = A \cos(\phi) \). Now solve for \( A \) and \( \phi \) in terms of \( K_1 \) and \( K_2 \); we get

\[ A = \sqrt{K_1^2 + K_2^2} \quad \text{and} \quad \tan(\phi) = \frac{K_1}{K_2}, \]  \hspace{1cm} (2)

where the quadrant in which \( \phi \) lies is determined by the signs of \( K_1 \) and \( K_2 \). This follows since \( \sin(\phi) = K_1/A \) so it has the same sign as \( K_1 \) and similarly \( \cos(\phi) \) has the same sign as \( K_2 \). Thus given values of coefficients \( C_1 \) and \( C_2 \) in Equation 1 we can compute the values of \( A \) and \( \phi \) from (2).

In the expression \( f(t) = A \sin(\omega t + \phi) \) \( \phi \) is called the **phase angle**.

A nonzero phase angle causes a \( f(t) \) to be a “shift” of the curve \( A \sin(\omega t) \). We see this as follows. Let \( f(t) = A \sin(\omega t + \phi) \). One complete sine wave of amplitude \( A \) is obtained as \( \omega t + \phi \) ranges from 0 to \( 2\pi \). Thus \( \omega t \) ranges from \(-\phi\) to \( 2\pi - \phi \) and so \( t \) ranges from \(-\phi/\omega\) to \( (2\pi - \phi)/\omega \). If \(-\phi/\omega < 0\), the shift is to the left. The number \(-\phi/\omega\) is often call the “phase shift” associated with function \( f(t) = A \sin(\omega t + \phi) \).

**Example 1.** Sketch the graph of \( f(t) = 3 \sin(2t - \pi/2) \).

The graph will be obtained by a phase shift of \( 3 \sin(2t) \) which has amplitude 3 and period \( \pi \). The shift will be \(-(-\pi/2)/2 = \pi/4\). Since the shift is \( > 0 \), it will be to the right. Basically we draw the graph of \( 3 \sin(2t) \), then shift all the points horizontally \( \pi/4 \) units to the right.

![Graph of Example 1](image)
Language that describes oscillations of various types

Sinusoidal function:

By a **sinusoidal function** we mean a curve described by $A \cos(\omega t)$, $B \sin(\omega t)$, or possibly a linear combination of these like $A \cos(\omega t) + B \sin(\omega t)$.

In terms of physical behavior such a function is periodic with a fixed amplitude.

Damped Sinusoids:

By a **damped sinusoidal function** we mean a curve described by $A \cos(\omega t)$, $B \sin(\omega t)$, or possibly a linear combination of these like $A \cos(\omega t) + B \sin(\omega t)$ multiplied by an exponential function $e^{-kt}$ where $k > 0$.

The physical behavior of such functions depends upon the type of exponential function that multiplies the sinusoid.
CASES:

#1. If the function like $A \cos(\omega t)e^{-kt}$ is such that the curve oscillates with decreasing amplitudes in successive oscillations we say that there is underdamping.

In a mass-spring system there is enough friction present to decrease the size of the vibrations as time progresses.

#2. If the function like $A \cos(\omega t)e^{-kt}$ is such that the curve does not oscillates we say that there is overdamping. The actual shape of the curve depend can vary a bit. A usual case is in the first frame below, but the second and third frame are also possibilities.

In a mass-spring system there is significant friction present to prevent vibrations as time progresses.
Next we consider the case of a mass-spring system in which there is friction (so it has damping) and there is an external driving force that is itself a sinusoid. This situation is called **FORCED, DAMPED OSCILLATIONS**.

**The story:** the mass-spring system vibrates with a sinusoidal motion at a certain frequency of motion, sometimes called the *natural frequency* of the system. The external force is a sinusoid with its own frequency of motion, sometimes called the *forcing frequency*.

**If the natural frequency is not the same as the forcing frequency,** then eventually the frequency of motion will be that of the forcing frequency. By eventually we mean that near the beginning of the motion of the system there maybe some erratic behavior (called the *transient state*), but eventually things will settle down and vibrate at the forcing frequency.

**If the natural frequency is the same as the forcing frequency,** then the behavior of the system is characterized by increasing amplitudes and can result in destructive consequences. Such systems are said to resonate or have resonant responses.
Envelope: A curve which touches every member of a family of curves or lines.

Example 2. The x- and y-axes are the envelope of the system of circles \((x-a)^2 + (y-a)^2 = a^2\).

Example 3. The envelope of the curve \(y = e^{-25t} \sin(t)\) is the curves \(y = e^{-25t}\) and \(y = -e^{-25t}\).