Section 9.6 The Exponential of a matrix

Key Terms:

- Power series
- Exponential of a matrix
- Truncation/Partial Sums
- Application to IVPs
- Generalized eigenvectors
To create a fundamental system of solutions for a system of DEs with constant coefficients we encountered a problem when we had repeated eigenvalues with geometric multiplicity strictly smaller than its algebraic multiplicity. Here we provide a strategy that will handle all cases.

One of our original IVPs was the simple linear IVP \( x' = ax, \ x(0) = x_0 \) which we know has the solution \( x(t) = x_0 e^{at} \).

For a system of linear DEs \( x' = Ax \) we pursued solutions of the form \( x(t) = e^{\lambda t}v \) where \((\lambda, v)\) was an eigen pair of matrix \( A \). To try to incorporate all the information in matrix \( A \) at one time we consider trying to formulate a solution of the system in the form 
\[
x(t) = e^{tA}v.
\]

Here \( v \) is just an \( n \)-vector.

At the moment, the term \( e^{tA} \), the exponential of a matrix, makes no sense. Our task, therefore, is to make sense of the exponential of a matrix, to learn as much about it as we can, to show that \( x(t) = e^{tA}v \) actually gives us a solution to the equation \( x' = Ax \), and to learn how to compute it.
The correct way to define the exponential of a matrix is not at all obvious. We will do so using a **power series**, in analogy to the power series for the exponential function,

\[
e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}a^k.
\]

**Definition:** The **exponential of the matrix** \(A\) **is** defined to be

\[
e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.
\]

The powers of matrix \(A\) make sense since \(A\) is square and the powers are repeated products of \(A\) with itself. By convention, \(A^0 = I\). Consequently, all of the terms make good sense. They are all \(n \times n\) matrices, so \(e^A\) is also an \(n \times n\) matrix, provided that the series converges.

Convergence of a series means that the sequence of partial sums converges. The components of the partial sums are very complicated expressions involving the entries of matrix \(A\). Convergence of the infinite series means that each component of the partial sum matrices converges. **It is a fact, although we will not prove it, that the series converges for every matrix** \(A\).
Let’s look at a simple special case.

**Example:** Show that the exponential of the diagonal matrix \( A = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \)
is the diagonal matrix

\[
e^A = \begin{bmatrix} e^r & 0 \\ 0 & e^s \end{bmatrix}
\]

Using the definition of the matrix exponential we have

\[
e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots
\]

Since \( A \) is a diagonal matrix, so are powers of \( A \) and we have

\[
e^A = I + \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}^3 + \cdots
\]

\[
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} r^2 & 0 \\ 0 & s^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} r^3 & 0 \\ 0 & s^3 \end{bmatrix} + \cdots
\]

\[
= \begin{bmatrix} 1 + r + (1/2!)r^2 + (1/3!)r^3 + \cdots & 0 \\ 0 & 1 + s + (1/2!)s^2 + (1/3!)s^3 + \cdots \end{bmatrix}
\]

\[
= \begin{bmatrix} e^r & 0 \\ 0 & e^s \end{bmatrix}
\]

Of course this extends to \( n \) by \( n \) diagonal matrices.
We have certain special cases: $e^{rI} = e^{rI}$ and $e^{0I} = e^{0I} = I$.

Unfortunately, there is no easy formula for the exponential of a matrix that is not diagonal.

We need to tie the matrix exponential to our IVP. Let $t$ be a real number then we have function

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots.$$ 

which has values that are $n$ by $n$ matrices.

But for $v$ an $n$-vector we have that function

$$e^{tA}v = v + tAv + \frac{t^2}{2!}A^2v + \frac{t^3}{3!}A^3v + \cdots,$$

has values which are $n$-vectors.

We need this because to obtain a fundamental set of solutions to a system of DEs we need to involve a basis for $\mathbb{R}^n$. 
Next we prove that the matrix exponential can be used to solve the IVP.

Suppose $A$ is an $n \times n$ matrix. We need to prove the following statements.

1. Then $\frac{d}{dt} e^{tA} = Ae^{tA}$

2. If $v \in \mathbb{R}^n$, the function $x(t) = e^{tA}v$ is the solution to the IVP $x' = Ax$ with $x(0) = v$.

**Proof of 1.** Just differentiate

\[
\frac{d}{dt} e^{tA} = \frac{d}{dt} \left( I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots \right) = A + \frac{t}{1!}A^2 + \frac{t^2}{2!}A^3 + \cdots = A \left( I + tA + \frac{t^2}{2!}A^2 + \cdots \right) = Ae^{tA}.
\]

**Proof of 2.** By part 1 we have

\[
\frac{d}{dt} x(t) = \frac{d}{dt} (e^{tA}v) = \frac{d}{dt} (e^{tA}) v = Ae^{tA}v = Ax(t).
\]

Then $x(0) = e^{0A}v = I v = v$. This shows we can solve the IVP if we can compute $e^{tA}v$. 
Notice that if most of the terms in the series
\[ e^{tA} = I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \cdots \]
are equal to the zero matrix, the infinite series becomes a finite sum.

For example, if \( A^2v = 0 \) and if \( p > 2 \), then \( A^p v = A^{p-2} A^2 v = A^{p-2} 0 = 0 \). Therefore, in the series we have

\[
e^{tA}v = v + tAv + \frac{t^2}{2!} A^2v + \frac{t^3}{3!} A^3v + \cdots
\]
\[
= v + tAv.
\]

When this happens we will say that the series for \( e^{tA}v \) truncates.

Here are some special cases:

1. If \( Av = 0 \), then \( e^{tA}v = v \) for all \( t \).
2. If \( A^2v = 0 \), then \( e^{tA}v = v + tAv \) for all \( t \).
3. More generally, if \( A^kv = 0 \), then

\[
e^{tA}v = v + tAv + \cdots + \frac{t^{k-1}}{(k-1)!} A^{k-1}v \quad \text{for all } t.
\]

Observe where the truncation occurs.

Note in the preceding that if vector \( v \) is in the null space of matrix \( A \) or a power of matrix \( A \) the series truncates to a finite sum (which looks like a matrix polynomial expression in \( t \) and powers of \( A \) times \( v \)). This property will be very useful.
Example: Let $A = \begin{bmatrix} -15 & 33 & 15 \\ -8 & 18 & 8 \\ -3 & 6 & 3 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $w = \begin{bmatrix} 9 \\ 4 \\ 0 \end{bmatrix}$.

Compute $e^{tA}v$ and $e^{tA}w$.

Claim: $v$ is in null space of $A$ $\implies$

So $e^{tA}v = v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Claim: $w$ is in null space of $A^2$ $\implies$

So $e^{tA}w = w + tAv$

1. If $Av = 0$, then $e^{tA}v = v$ for all $t$.
2. If $A^2v = 0$, then $e^{tA}v = v + tAv$ for all $t$. 

```matlab
>> A,A*[1 0 1]'
A =
-15 33 15
-8 18 8
-3 6 3
ans =
0
0
0

>> A^2,A^2*[9 4 0]'
an =
-84 189 84
-48 108 48
-12 27 12
ans =
0
0
0
```
Properties of the matrix exponential:

1. If $A$ and $B$ are $n \times n$ matrices, then $e^{A+B} = e^A e^B$ if and only if $AB = BA$.

2. If $A$ is an $n \times n$ matrix, then $e^A$ is a nonsingular matrix whose inverse is $e^{-A}$.

The proofs just use the definition of the matrix exponential and #2 uses #1.

Recall that we need a basis for $\mathbb{R}^n$ to be able to obtain a fundamental set of solutions to a system of DEs $x' = Ax$. The basis we want consists of eigenvectors if $A$ is diagonalizable. Thus we will need the eigenvalues and eigenvectors of $A$. If $A$ is defective we will need enough generalized eigenvectors to combine with the eigenvectors to obtain the basis. Recall that eigenvectors belong to the $\text{ns}(A - \lambda I)$ and the generalized eigenvectors are in the null spaces of powers of $(A - \lambda I)$. So we need to see how to get $A - \lambda I$ involved.
For the following you can just think of $\lambda$ as a scalar; we will see what happens when it is an eigenvalue of $A$.

We want to increase our capability of computing $e^{tA}v$. Notice that if $\lambda$ is a number, then

$$tA = \lambda t \mathbf{I} + t[A - \lambda \mathbf{I}].$$

Forming the matrix exponential of each side and simplifying we get

$$e^{tA} = e^{\lambda t \mathbf{I} + t[A - \lambda \mathbf{I}]} = e^{\lambda t \mathbf{I}} e^{t[A - \lambda \mathbf{I}]}.$$

Matrix $\lambda t \mathbf{I}$ is diagonal so we can compute $e^{\lambda t \mathbf{I}} = e^{\lambda t} \mathbf{I}$ so we have

$$e^{tA} = e^{\lambda t} \mathbf{I} e^{t[A - \lambda \mathbf{I}]} = e^{\lambda t} e^{t[A - \lambda \mathbf{I}]}$$

(we multiplied by the identity matrix). This is an ordinary function.

Finally for a vector $v$ we get $e^{tA}v = e^{\lambda t} e^{t[A - \lambda \mathbf{I}]}v$. $\Leftarrow$ We need this to prove the following.

Next we relate things to eigenvalues and eigenvectors. Suppose $(\lambda, v)$ is an eigenpair of $A$.

1. Since $[A - \lambda \mathbf{I}]v = \mathbf{0}$, then $e^{tA}v = e^{\lambda t}v$ for all $t$. $\Leftarrow$ note the switch from $A$ to $\lambda$; explain.

2. If $[A - \lambda \mathbf{I}]^2v = \mathbf{0}$, then $e^{tA}v = e^{\lambda t} (v + t[A - \lambda \mathbf{I}]v)$ for all $t$. $\Leftarrow v$ is a generalized eigenvector of order 2.

3. More generally, if $k$ is a positive integer and $[A - \lambda \mathbf{I}]^k v = \mathbf{0}$, then

$$e^{tA}v = e^{\lambda t} \left( v + t[A - \lambda \mathbf{I}]v + \cdots + \frac{t^{k-1}}{(k-1)!} [A - \lambda \mathbf{I}]^{k-1}v \right) \text{ for all } t.$$ $\Leftarrow v$ is a generalized eigenvector of order $k$.  

Used $e^{A+B} = e^A e^B$ if and only if $AB = BA$.  

This is an ordinary function.
Let's put all the pieces together to find a fundamental set of solutions to \( x' = Ax \)
where
\[
A = \begin{bmatrix}
-1 & 2 & 1 \\
0 & -1 & 0 \\
-1 & -3 & -3
\end{bmatrix}
\]

The eigenvalues are -1, -2, -2 and the eigen pairs are \( -1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \) and \( -2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \).

From the eigen pairs we have a pair of linearly independent solutions
\[
x_1(t) = e^{tA} \mathbf{v}_1 = e^{-t} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}
x_2(t) = e^{tA} \mathbf{v}_2 = e^{-2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\]

Next we find a generalized eigenvector corresponding to -2.

The general solution of the null space for \((A -(-2)I)^2\) is \(\begin{bmatrix} r \\ 0 \\ s \end{bmatrix}\).

So one generalized eigenvector is \( \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\).

We used \( e^{tA} \mathbf{v} = e^{\lambda t} \mathbf{v} \) since we have eigenpairs.

We used \( e^{tA} \mathbf{v} = e^{\lambda t} (\mathbf{v} + t[A - \lambda I] \mathbf{v}) \) since \([A - \lambda I]^2 \mathbf{v}_3 = 0\).

so we have \( x_3(t) = e^{tA} \mathbf{v}_3 = e^{-2t} (\mathbf{v}_3 + t[A + 2I] \mathbf{v}_3) \).
\[ x_1(t) = e^{tA}v_1 = e^{-t} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \]
\[ x_2(t) = e^{tA}v_2 = e^{-2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \]
\[ x_3(t) = e^{tA}v_3 = e^{-2t} \left( v_3 + t[A + 2I]v_3 \right) = e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = e^{-2t} \begin{bmatrix} 1 + t \\ 0 \\ -t \end{bmatrix} \]

This is our fundamental set of solutions. So the general solution is

\[ x(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + C_3 e^{-2t} \begin{bmatrix} 1 + t \\ 0 \\ -t \end{bmatrix} \]
**Example:** Find a fundamental set of solutions for \( y' = Ay \), where

\[
A = \begin{bmatrix}
7 & 5 & -3 & 2 \\
0 & 1 & 0 & 0 \\
12 & 10 & -5 & 4 \\
-4 & -4 & 2 & -1
\end{bmatrix}
\]

Find eigen information.

\[
>> [v,d]=eig(A)
\]

\[
v =
\begin{bmatrix}
-0.4082 & 0.4472 + 0.0000i & 0.4472 - 0.0000i & 0.1945 \\
0 & 0 & 0 & 0.4062 \\
-0.8165 & 0.8944 & 0.8944 & 0.7951 \\
0.4082 & 0.0000 - 0.0000i & 0.0000 + 0.0000i & -0.4062
\end{bmatrix}
\]

\[
d =
\begin{bmatrix}
-1.0000 & 0 & 0 & 0 \\
0 & 1.0000 + 0.0000i & 0 & 0 \\
0 & 0 & 1.0000 - 0.0000i & 0 \\
0 & 0 & 0 & 1.0000
\end{bmatrix}
\]

**Eigenvalues:** -1, 1, 1, 1

**Eigenvectors:**
looking at the data we have 3 linearly independent eigenvectors; we need a generalized eigenvector for \( \lambda = 1 \)

The generalized eigenvector will be of order 2; thus it is in the null space of \((A - \lambda I)^2\).
Determining the generalized eigenvector:

The general form for vectors in the null space of \((A - \mathbf{I})^3\) is

\[
\begin{bmatrix}
-r + 0.5s - 0.5t
\end{bmatrix}
\begin{bmatrix}
r \\ s \\ t
\end{bmatrix}
\]

One vector is obtained by setting
\(r = 1, s = t = 1\)
which gives vector
\[
v_4 = \begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

Recall we have eigen pairs

\[
\begin{bmatrix}
-1 \\ 0 \\ 1
\end{bmatrix}, \begin{bmatrix}
1/2 \\ 0 \\ 0
\end{bmatrix}, \begin{bmatrix}
1/2 \\ 0 \\ 1
\end{bmatrix}
\]

Our fundamental set:

\[
x_1(t) = e^{tA}v_1 = e^{-t} \begin{bmatrix}
-1 \\ 0 \\ 1
\end{bmatrix}
x_2(t) = e^{tA}v_2 = e^{t} \begin{bmatrix}
1/2 \\ 0 \\ 0
\end{bmatrix}
x_3(t) = e^{tA}v_3 = e^{t} \begin{bmatrix}
1/2 \\ 0 \\ 1
\end{bmatrix}
x_4(t) = e^{tA}v_4 = e^{t} (v_4 + t[A - \mathbf{I}]v_4)
\]

\[
= e^{t} \begin{bmatrix}
0 \\ 0 \\ 1 \\
1 \\ 0 \\ 1
\end{bmatrix} + t \begin{bmatrix}
6 \\ 12 \\ -4 \\
5 \\ -6 \\ -4 \\
-3 \\ 4 \\ 2
\end{bmatrix} \begin{bmatrix}
0 \\ 0 \\ 1
\end{bmatrix}
\]

\[
= e^{t} \begin{bmatrix}
-t \\ 0 \\ -2t
\end{bmatrix} + t \begin{bmatrix}
-t \\ 0 \\ -1
\end{bmatrix}
\]

\[
= e^{t} \begin{bmatrix}
-t \\ 0 \\ -1 - 2t
\end{bmatrix}
\]
So the general solution of the system is

\[ x(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + C_3 e^t \begin{bmatrix} 1/2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + C_4 e^t \begin{bmatrix} -t \\ 0 \\ 1-2t \\ 1 \end{bmatrix} \]
**Example:** Here we consider repeated complex roots.

Find a fundamental system of solutions for the system $x' = Ax$, where

\[
A = \begin{bmatrix}
6 & 6 & -3 & 2 \\
-4 & -4 & 2 & 0 \\
8 & 7 & -4 & 4 \\
1 & 0 & -1 & -2
\end{bmatrix}
\]

**Eigenvalues:** repeated values of $-1 \pm i$

**Eigen pairs:**

\[
\begin{align*}
-1 + i, w_1 &= \begin{bmatrix} 1+i \\ 0 \\ 2+2i \\ -1 \end{bmatrix} \\
-1 - i, w_3 &= \begin{bmatrix} 1-i \\ 0 \\ 2-2i \\ -1 \end{bmatrix}
\end{align*}
\]

**Strategy:** Find a generalized eigenvector corresponding to eigenvalue $-1 + i$. Use the matrix exponential to find the two complex solutions. Then use complex conjugates and find real and complex parts to get four real functions as a fundamental set.
For the first eigenpair we have complex solution

\[ z_1(t) = e^{tA} w_1 = e^{(-1+i)t} w_1 = e^{(-1+i)t} \begin{bmatrix} 1 + i \\ 0 \\ 2 + 2i \\ -1 \end{bmatrix} \]

For a generalized eigenvector we look in the null space of \((A-(-1+i)I)^2\).

\[
\begin{array}{c}
>> \text{rref}([\text{syms}('A')-(-1+i)*eye(4)]^2 [0 0 0 0]')
\end{array}
\]

\[
\text{ans} =
\begin{bmatrix}
1, 0, -7/8 - i/8, -1/2 - i, 0 \\
0, 1, 1/2, 1 + i, 0 \\
0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0
\end{bmatrix}
\]

The general solution has the form

\[
\begin{bmatrix}
(7/8 + 1/8 i)r + (1/2 + i)s \\
-1/2r - (1+i)s \\
r \\
s
\end{bmatrix}
\]

Choosing \( r = 0, s = 2 \) we get generalized eigenvector

\[
w_2 = \begin{bmatrix} 1 + 2i \\ -2 - 2i \\ 0 \\ 2 \end{bmatrix}
\]
For eigenvalue \(-1 + i\) we have generalized eigenvector

The corresponding solution is

\[
\begin{pmatrix}
1+2i \\
-2-2i \\
-2 \\
2
\end{pmatrix}
\]

Assume that \(A\) and \(w_2\) have been entered into MATLAB.

```matlab
>> AA=sym(A);ww2=sym(w2);vv=(ww2+sym('t')*(AA-sym('1+i')*sym(eye(4))))*ww2

vv =

t*(1 + i) + 1 + 2*i
   - 2 - 2*i
   t*(2 + 2*i)
   2 - t
```

\(\Rightarrow\) Simplify \(vv\) to get the vector.
Let's find the real part of \( z_1(t) = e^{tA} w_1 = e^{(-1+i)t} w_1 = e^{(-1+i)t} \left( \begin{array}{c} 1+i \\ 0 \\ 2+2i \\ -1 \end{array} \right) \)

\[
e^{(-1+i)t} = e^{-t} (\cos(t) + i \sin(t))
\]

\[
= e^{-t} \cos(t) - e^{-t} \sin(t) + i \left( e^{-t} \cos(t) + e^{-t} \sin(t) \right)
\]

The real part is

\[
f = e^{-t} \cos(t) - e^{-t} \sin(t)
\]

It should be that \( f' = Af \) see MATLAB code next slide.
>> f=[sym('exp(-t)*(cos(t)-sin(t))');sym('0');sym('exp(-t)*(2*cos(t)-2*sin(t))');sym('exp(-t)*(-cos(t))')]
f =
   exp(-t)*(cos(t) - sin(t))
   0
   exp(-t)*(2*cos(t) - 2*sin(t))
   -exp(-t)*cos(t)
>> g=diff(f);
>> gg=AA*f;
>> g-gg
ans =
   2*exp(-t)*cos(t) - 7*exp(-t)*(cos(t) - sin(t)) + 3*exp(-t)*(2*cos(t) - 2*sin(t)) - exp(-t)*(cos(t) + sin(t))
   4*exp(-t)*(cos(t) - sin(t)) - 2*exp(-t)*(2*cos(t) - 2*sin(t))
   4*exp(-t)*cos(t) - 8*exp(-t)*(cos(t) - sin(t)) + 3*exp(-t)*(2*cos(t) - 2*sin(t)) - exp(-t)*(2*cos(t) + 2*sin(t))
   exp(-t)*(2*cos(t) - 2*sin(t)) - exp(-t)*cos(t) - exp(-t)*(cos(t) - sin(t)) + exp(-t)*sin(t)
>> ans1=simplify(ans)
ans1 =
   0
   0
   0
   0
The other eigenvalue is the conjugate of $-1 + i$ so the conjugate of $w_1$ and the conjugate of $w_2$ are an eigenvector and generalized eigenvector corresponding to eigenvalue $-1 - i$. Then the conjugates of $z_1$ and $z_2$ are corresponding solutions.

To find real solutions, we use the real and imaginary parts of the complex solutions. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $x_1$, $x_2$, $y_1$, and $y_2$ are the four needed real solutions.

Repeated complex eigenvalues require careful work to obtain the fundamental set.