Section 9.3 Phase Plane Portraits (for Planar Systems)

Key Terms:

• Equilibrium point of planer system $y' = Ay$
  o Equilibrium solution
• Exponential solutions
  o Half-line solutions
• Unstable solution
• Stable solution
• Six important cases for portraits
  ❖ Real Eigenvalues
    o Saddle point
    o Nodal sink
    o Nodal source
  ❖ Complex Eigenvalues
    o Center
    o Spiral sink
    o Spiral source
• Direction of rotation
We know how to solve linear planar systems with constant coefficients, so now we investigate what the solutions look like. There is a variety of different cases, with different behaviors. **We will examine the six most important cases here.**

We consider a planar autonomous system $y' = Ay$ with $\lambda^2 - T\lambda + D = 0$ where $D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$, and $T = \text{tr}(A) = a_{11} + a_{22}$.

Since the system is autonomous, it is natural to use the phase plane to visualize the solutions. Recall in particular, that the uniqueness theorem implies that solution curves cannot intersect.

For any autonomous system (linear or nonlinear) $x' = f(x)$, a vector $x_0$ for which $f(x_0) = 0$ is called an **equilibrium point**. The function $x(t) = x_0$ satisfies the equation and is called an **equilibrium solution**.

For the linear homogeneous system $y' = Ay$ the equilibrium points are those points $v \in \mathbb{R}^2$ where $Av = 0$. Thus, the set of equilibrium points is just the nullspace of $A$. We will find that the solutions of system $y' = Ay$ have strong connections to the equilibrium points of the system. **In most of the cases that we will examine, $A$ is nonsingular, so the origin $0$ is the only equilibrium point.** There will be times when $A$ is singular, in which case the nullspace is a line in $\mathbb{R}^2$, or all of $\mathbb{R}^2$, and every point in the nullspace is an equilibrium point.
The case of real eigenvalues

For this to be true, we must have $T^2 - 4D > 0$. The eigenvalues are

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

Hence the eigenvalues of $A$ will be real and distinct with $\lambda_1 < \lambda_2$.

Thus the general solution is

$$y(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2,$$

where $C_1$ and $C_2$ are arbitrary constants and $v_1$ and $v_2$ are eigenvectors associated with $\lambda_1$ and $\lambda_2$, respectively. **Particular solutions** are captured by assigning various values to the constants $C_1$ and $C_2$. 

Exponential solutions

Two particular solutions are especially noteworthy. These occur when one, of the constants $C_1$ and $C_2$ is equal to zero. These are the solutions and

$$C_1 e^{\lambda_1 t} v_1 \quad \text{and} \quad C_2 e^{\lambda_2 t} v_2,$$

the so-called exponential solutions.

To simplify the notation, let’s suppose that $\lambda$ is an eigenvalue of the matrix $A$, and $v$ is an associated eigenvector. Then the exponential solutions have the form

$$y(t) = Ce^{\lambda t} v$$

If $C > 0$, $y(t)$ is a positive multiple of $v$, and if $C < 0$, $y(t)$ is a positive multiple of $-v$.

If $\lambda > 0$, the exponential $e^{\lambda t}$ increases from 0 to $\infty$ as $t$ increases from $-\infty$ to $\infty$, while if $\lambda < 0$ it decreases from $\infty$ to $-\infty$. In either case, $y(t)$ traces out the half-line consisting of positive multiples of $Cv$.

Thus there are precisely two solution curves in the phase plane depending on the sign of the constant $C$. Since these are half-lines, we will sometimes refer to exponential solutions as half-line solutions.

Next we need to consider the movement of a solution with respect to equilibrium point at the origin.
For a positive eigenvalue: $e^{\lambda t}$ increases. The **exponential solutions** tend **away** from the equilibrium point at the origin as $t$ increases and tend to the origin as $t$ decreases to $-\infty$. Solutions with this property are called **unstable solutions**.

For a negative eigenvalue: $e^{\lambda t}$ decreases to 0 as $t \to \infty$. Thus the **exponential solution approaches** the equilibrium point at the origin as $t \to \infty$. Such solutions are said to be **stable**.
Consider the general situation in which both constants \( C_1 \) and \( C_2 \) are nonzero. So we are considering the super position of two exponential solutions.

\[
y(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2
\]

The phase portrait varies depending on the signs of the eigenvalues.

**Case \( \lambda_1 < 0 < \lambda_2 \):** One exponential solution will be stable and the other unstable. This type of behavior implies solutions can move towards equilibrium but then move away. This is the type of behavior describing a *saddle point*.

**Example:** \( \mathbf{y}' = A\mathbf{y} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \mathbf{y} \)  
Eigen pairs \( -3, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) \( 3, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \)

Fundamental set \( e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \)

Half lines:

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow y_2 = -y_1
\]
\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow y_2 = (1/2)y_1
\]
Half lines are shown together with some particular solutions.

The exponential, or half-line, solutions separate the plane into four regions in which the solution curves have different behavior. For this reason, these curves (half-lines) are called *separatrices*.

Saddle surface.
**Case** $\lambda_1 < \lambda_2 < 0$: Both exponential solutions will be stable. The exponential solutions will decay to the origin (the equilibrium point) along the half-lines. A distinguishing characteristic of a planar linear system with two negative eigenvalues is that all solution curves approach the origin as $t \to \infty$ with a well-defined tangent line. This is the type of behavior describing a **nodal sink**.

**Example:** \[ y' = Ay = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix} y \]

Eigen pairs $-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $-4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Fundamental set $e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Half lines:

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow y_2 = -y_1
\]

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow y_2 = y_1
\]
Half lines are shown together with some particular solutions.

Note how the particular solutions decay to the origin in a direction parallel to the exponential solutions (half lines).
Case $0 < \lambda_1 < \lambda_2$: Both exponential solutions will be unstable. The exponential solutions will emanate away from the origin (the equilibrium point) along the half-lines. A distinguishing characteristic of a planar linear system with two positive eigenvalues is that all solution curves approach the origin as $t \to -\infty$ with a well-defined tangent line. (Equivalently as $t \to \infty$ the solution emanate from the origin, increasing without bound.) This is the type of behavior describing a **nodal source**.

**Example:** \[ y' = Ay = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} y \]

Eigen pairs: \[
\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}
\]

Fundamental set \[
e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\]

Half lines:

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow y_2 = -y_1
\]

\[
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow y_2 = y_1
\]
Half lines are shown together with some particular solutions.

Note how the particular solutions emanate from the origin.

Look at the small gray arrows.
Complex eigenvalues:
In the next three cases we study situations where the eigenvalues are complex. We saw that a complex eigenvalue \( \lambda = \alpha + i\beta \) and its associated eigenvector \( w = v_1 + iv_2 \) lead to the general solution

\[
y(t) = C_1 e^{\alpha t} (\cos \beta t \, v_1 - \sin \beta t \, v_2) + C_2 e^{\alpha t} (\sin \beta t \, v_1 + \cos \beta t \, v_2)
\]

**Case eigenvalues are purely imaginary:** In this case \( \alpha = 0 \) so the solution involves just sine and cosine terms which are periodic with period \( T = \frac{2\pi}{|\beta|} \). The vector-valued function \( y(t) \) has the same property, so the solution trajectory is a closed curve orbiting about the origin. This is the type of behavior describing a center.

**Example:** \( \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \)

One eigenpair is \( \lambda = 2i, w = \begin{bmatrix} 1 \\ i \end{bmatrix} \)

Therefore, we have the complex-valued exponential solution \( z(t) = e^{\lambda t}w \). To find real-valued solutions we compute the real and imaginary parts of \( z(t) \). Using Euler’s formula we find

\[
z(t) = \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + i \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}
\]

So the real and imaginary parts are a fundamental set. We can show that the solution curves are circles in this case by showing that the components of \( y(t) = (y_1(t), y_2(t))^T \) satisfy \( y_1(t)^2 + y_2(t)^2 = \text{constant} \).
The distinguishing characteristic of this type of equilibrium point is that it is surrounded by closed solution curves. An equilibrium point for any planar system, linear or nonlinear, that has this property is called a **center**. Planar linear systems with purely imaginary eigenvalues have centers at the origin.

Not all centers have solution curves that are circles. This is not even true for linear systems. We can prove that, for a linear center, the orbits are similar ellipses centered at the origin. For example, the system

\[
y' = Ay = \begin{bmatrix} 4 & -10 \\ 2 & -4 \end{bmatrix} y
\]

has eigenvalue 2i and associated eigenvector \((2+i, 1)^T\), so the equilibrium point is still a center, but the solution curves are ellipses.
Case real part of the eigenvalues is negative: In this case $\alpha < 0$ so $e^{\alpha t} \to 0$ as $t \to \infty$. The solution curves circle the origin but are drawn towards it at the same time, resulting in a spiral motion. Since all solution curves spiral to the equilibrium point at the origin, all solutions are stable. This type of behavior characterizes a **spiral sink**.

**Example:**

$$\dot{y} = Ay = \begin{bmatrix} 1 & -4 \\ 2 & -3 \end{bmatrix} y$$

One eigenpair is $\lambda = -1 + 2i, \mathbf{w} = \begin{bmatrix} 2 \\ 1 - i \end{bmatrix}$

In the phase portrait shown, a number of solution trajectories swirl about the origin, decaying to $\mathbf{0}$ as $t \to \infty$. 

![Phase portrait](image-url)
**Case real part of the eigenvalues is positive:** In this case $\alpha > 0$ so $e^{\alpha t} \to \infty$ as $t \to \infty$. The solution curves circle the origin but the amplitude of the oscillation increases, resulting in an **outward spiral motion**. Since all solution curves spiral away from the equilibrium point at the origin, all solutions are unstable. This type of behavior characterizes a **spiral source**.

**Example:** \( y' = Ay = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} y \)  
One eigenpair is \( \lambda = 1 + i, w = \begin{bmatrix} 0.5 + 0.5i \\ 1 \end{bmatrix} \)
Characterization of equilibrium points

The different (common) phase portraits for a linear, planar system \( \mathbf{y}' = A\mathbf{y} \).

- **Saddle point.** A has two real eigenvalues, one positive and one negative.
- **Nodal sink.** A has two negative real eigenvalues.
- **Nodal source.** A has two positive real eigenvalues.
- **Center.** A has two complex-conjugate eigenvalues with zero real parts.
- **Spiral sink.** A has two complex-conjugate eigenvalues with negative real parts.
- **Spiral source.** A has two complex-conjugate eigenvalues with positive real parts.
In our examples we used a numerical solver to see the phase plane. If that is not available then one quick way to determine the rotation direction is simply to compute one vector in the vector field determined by the right-hand side of system. Use a convenient point line \((1, 0)\) and compute

\[
A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}
\]

Then connect \((1,0)\) to \((a_{11}, a_{21})\) to get the orientation for the direction of rotation about the equilibrium point.

If \(a_{21} > 0\), this vector points into the upper half-plane, indicating that the rotation is counterclockwise. On the other hand, if \(a_{21} < 0\), this vector points into the lower half-plane, indicating that the rotation is clockwise.