Section 9.2a Planar Systems Derived from Linear Second Order DEs

Key Terms:

• Second-order differential equation

• Second-order linear differential equation

• Forcing term or external force

• Homogeneous second order linear DE

• Second order linear homogeneous DEs with constant coefficients

• Higher order linear homogeneous DEs with constant coefficients
A second-order differential equation is an equation involving the independent variable t and an unknown function y along with its first and second derivatives. We will assume it is possible to solve for the second derivative, in which case the equation has the form

\[ y'' = f(t, y, y'). \]

A solution to such an equation is a twice continuously differentiable function y(t) such that

\[ y''(t) = f(t, y(t), y'(t)). \]

A second-order linear differential equation has the special form

\[ y'' + p(t)y' + q(t)y = g(t). \]

The coefficients p, q, and g can be arbitrary functions of the independent variable t, but y'', y', and y must all appear to first “power”.

The function \( g(t) \) on the right side of equation is called the forcing term since it often arises from an external force. If the forcing term is equal to 0, the resulting equation is said to be homogeneous second order linear DE. It has the form

\[ y'' + p(t)y' + q(t)y = 0. \]

If p and q are constants then the DE has the form \( y'' + py' + qy = 0 \). Such DEs are called second order linear homogeneous DEs with constant coefficients. This class of DEs is easily solved.
Recall that the DE \( ay'' + by' + cy = 0 \) can be written as a system of first order DES as follows:

Let \( u_1 = y \) and \( u_2 = y' \), then we differentiate both of these expressions

\[
\begin{align*}
u_1' &= y' = u_2 \\
u_2' &= y'' = -\frac{b}{a}y' - \frac{c}{a}y = -\frac{b}{a}u_2 - \frac{c}{a}u_1
\end{align*}
\]

Then we get planar system

\[
\begin{align*}
u_1' &= u_2 \\
u_2' &= -\frac{c}{a}u_1 - \frac{b}{a}u_2
\end{align*}
\]

which has matrix form

\[
\begin{bmatrix}
u_1' \\
u_2'
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
\frac{-c}{a} & \frac{-b}{a}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

The characteristic equation is

\[
p(\lambda) = \det
\begin{bmatrix}
0 & 1 \\
\frac{-c}{a} & \frac{-b}{a}
\end{bmatrix}
- \lambda
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
= \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0
\]

which we can express as

\[
p(\lambda) = a\lambda^2 + b\lambda + c = 0
\]

Since the characteristic equation is a quadratic equation, its roots are given by

the quadratic formula

\[
\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
λ = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}

Looking at the discriminant \(b^2 - 4ac\), we see that there are three cases to consider:

1. two distinct real roots if \(b^2 - 4ac > 0\);
2. two distinct complex roots if \(b^2 - 4ac < 0\);
3. one repeated real root if \(b^2 - 4ac = 0\).

We know how to address each of these cases. We find a fundamental set of solutions to system

\[
\begin{bmatrix}
  u' \\
  u'
\end{bmatrix} = \begin{bmatrix}
  0 & 1 \\
  -\frac{c}{a} & -\frac{b}{a}
\end{bmatrix} \begin{bmatrix}
  u \\
  u
\end{bmatrix}
\]

then the solution of DE \(ay'' + by' + cy = 0\) is \(u_1\).

1. two distinct real roots \(\lambda_1\) and \(\lambda_2\); fundamental set is \(e^{\lambda_1 t}, e^{\lambda_2 t}\)

2. Two complex roots if \(\alpha \pm \beta i, \beta \neq 0\); fundamental set is \(e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t)\)

3. one repeated real root if \(\lambda_1 = \lambda_2\); fundamental set is \(e^{\lambda_1 t}, te^{\lambda_1 t}\)

The general solution \(u_1\) is a linear combination of the functions in the fundamental set with arbitrary coefficients.

If we have IVP \(ay'' + by' + cy = 0, y(t_0) = y_0, y'(t_0) = yp_0\) use the general solution and apply the initial conditions to find the constants \(C_1\) and \(C_2\).
What is easy about solving second order linear homogeneous DEs with constant coefficients is that we need NOT convert to a system and find eigen information. We outline the procedure next.

1. For $ay'' + by' + cy = 0$ find the roots of $p(\lambda) = a\lambda^2 + b\lambda + c = 0$.

2. Based on the roots construct the corresponding fundamental set of solutions as one of the following:

   $e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t), e^{\lambda_1 t}, te^{\lambda_1 t}$

3. The general solution $y(t)$ is a linear combination of the functions in the fundamental set with arbitrary coefficients $C_1$ and $C_2$.

4. If we have an IVP in the general solution apply the initial conditions to find the constants $C_1$ and $C_2$. 
Example:

\[ y'' - 3y' - 10y = 0, \quad y(0) = -1, \quad y'(0) = 0 \]

Fundamental set: \( e^{-2t}, e^{5t} \)

General solution: \( y = C_1 e^{-2t} + C_2 e^{5t} \) \( \Rightarrow \) \( y' = -2 C_1 e^{-2t} + 5 C_2 e^{5t} \)

Linear system \( \Rightarrow \)
\[
\begin{align*}
C_1 + C_2 &= -1 \\
-2C_1 + 5C_2 &= 0
\end{align*}
\]

Solution: \( y = \frac{-5}{7} e^{-2t} - \frac{2}{7} e^{5t} \)

What is \( \lim_{t \to 0} y(t) \) ?

What is \( \lim_{t \to \infty} y(t) \) ?

Example:

\[ y'' + 6y' + 9y = 0, \quad y(0) = -1, \quad y'(0) = 1 \]

Fundamental set: \( e^{-3t}, te^{-3t} \)

General solution: \( y = C_1 e^{-3t} + C_2 te^{-3t} \) \( \Rightarrow \) \( y' = -3 C_1 e^{-3t} + C_2 e^{-3t} - 3 C_2 t e^{-3t} \)

Linear system \( \Rightarrow \)
\[
\begin{align*}
C_1 &= -1 \\
-3C_1 + C_2 &= 1
\end{align*}
\]

Solution: \( y = -e^{-3t} - 2t e^{-3t} \)

What is \( \lim_{t \to \infty} y(t) \) ?
Example:
\[ y'' + 0.4y' + 1.04y = 0, \ y(0) = 0.5, \ y'(0) = 1 \]
Fundamental set \( e^{-t/5}\cos(t), \ e^{-t/5}\sin(t) \)

General solution: \( y = C_1 e^{-t/5}\cos(t) + C_2 e^{-t/5}\sin(t) \)

\[
y' = -\frac{C_1}{5} e^{-t/5}\cos(t) - C_1 e^{-t/5}\sin(t) - \frac{C_2}{5} e^{-t/5}\sin(t) + C_2 e^{-t/5}\cos(t)
\]

Linear system \( \Rightarrow \)
\[
-C_1 + C_2 = 1
\]
\( C_1 = 0.5 \)
\( \frac{-C_1}{5} + C_2 = 1 \) \( C_1 = -0.5, \ C_2 = 11/10 \)

Solution: \( y = -0.5 e^{-t/5}\cos(t) + 11/10 e^{-t/5}\sin(t) \)

What is \( \lim_{t \to \infty} y(t) \) ?

What does the graph of the solution look like?

Sinusoid of decreasing amplitude.
Higher order linear homogeneous DEs with constant coefficients

The forms of fundamental sets of solutions generalize to higher order DEs. We illustrate this by examples.

**Example:** Determine the general solution to $y^{\prime\prime\prime} - y^{\prime\prime} - y^{\prime} + y = 0$.

Find the characteristic equation and compute its roots.

We have $p(\lambda) = \lambda^3 - \lambda^2 - \lambda + 1 = 0$.
This factors as $\lambda^2(\lambda - 1) - (\lambda - 1) = (\lambda - 1)(\lambda^2 - 1) = 0$ so $\lambda = 1, 1, -1$.

The fundamental set is $e^{-t}, e^t, te^t$.

Thus the general solution is $y(t) = C_1e^{-t} + C_2e^t + C_3te^t$
**Example:** Determine the general solution to $y^{(4)} - 4y''' + 4y'' = 0$.

Find the characteristic equation and compute its roots.

We have $p(\lambda) = \lambda^4 - 4\lambda^3 + 4\lambda^2 = 0$.

This factors as $\lambda^2(\lambda^2 - 4\lambda + 4) = 0$ so $\lambda = 0, 0, 2, 2$

The fundamental set is $e^0, te^0, e^{2t}, te^{2t}$.

Thus the general solution is $y(t) = C_1 + C_2 t + C_3 e^{2t} + C_4 te^{2t}$.