Section 9.2 Planar Systems Part 1 Complex Case

Key Terms:

• Planar systems
• Characteristic equation for planar systems
  o Complex roots
• Euler’s Formula
• Linear Independent solutions
The characteristic polynomial of a 2 by 2 matrix

Let $A$ be the general real $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Next we expand the $\det(A - \lambda I)$ and obtain an equation in $\lambda$ to use to determine the eigenvalues of $A$.

$$
\det(A - \lambda I) = \det\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det\left( \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \right) = (a - \lambda)(d - \lambda) - bc
$$

$$
= ad - a\lambda - d\lambda + \lambda^2 - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0
$$

Thus we have a compact expression for the characteristic equation in terms of the trace and determinant of matrix $A$.

$$
\det(A - \lambda I) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0
$$

**Definition:** If $A$ is a square matrix, then the trace of matrix $A$, denoted $\text{tr}(A)$, is the sum of the diagonal entries.

If $A$ is $n \times n$ then $\text{tr}(A) = \sum_{j=1}^{n} a_{jj}$
\[
\det(A - \lambda I) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0
\]

Let \( D = \det(A) \) and \( T = \text{tr}(A) \), then the characteristic equation of the planar system becomes

\[
\lambda^2 - T\lambda + D = 0.
\]

The eigenvalues of \( A \) are the roots of the characteristic polynomial and are given by

\[
\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}
\]

There are three cases we must consider:

1. two distinct real roots (when \( T^2 - 4D > 0 \)) \( \Leftarrow \) this one is known
2. two complex conjugate roots (when \( T^2 - 4D < 0 \))
3. one real root of multiplicity 2 (when \( T^2 - 4D = 0 \))
Complex roots $T^2 - 4D < 0$.

The roots of the characteristic equation are the complex conjugates

$$
\lambda = \frac{T + i\sqrt{4D - T^2}}{2} \quad \text{and} \quad \bar{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}.
$$

Suppose we have a matrix $A$ with real entries, and complex conjugate
eigenvalues $\lambda$ and $\bar{\lambda}$. Suppose that $w$ is an eigenvector associated with $\lambda$.
Then its complex conjugate $\bar{w}$ is an eigenvector corresponding to $\bar{\lambda}$.

Proof: $Aw = \lambda w$  Take the conjugate of both sides and use that $A$ has only real entries.

\[
\bar{A} \bar{w} = \bar{\lambda} \bar{w} \quad \Rightarrow \quad A \bar{w} = \bar{\lambda} w \quad \Rightarrow \quad A \bar{w} = \bar{\lambda} \bar{w}
\]

Theorem:

Suppose that $A$ is a $2 \times 2$ matrix with complex conjugate eigenvalues $\lambda$ and $\bar{\lambda}$.
Suppose that $w$ is an eigenvector associated with $\lambda$. Then the general solution to the
system $y' = Ay$ is

$$
y(t) = C_1 e^{\lambda t} w + C_2 e^{\bar{\lambda} t} \bar{w},
$$

where $C_1$ and $C_2$ are arbitrary constants.
**Example:** Find a fundamental set of solutions to $y' = Ay$ where $A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$

>> [v,d]=eig(A)  

MATLAB answers.  

$v = \begin{bmatrix} 0.4082 - 0.4082i & 0.4082 + 0.4082i \\ 0.8165 & 0.8165 \end{bmatrix}$  

d = \begin{bmatrix} 1.0000 + 1.0000i & 0 \\ 0 & 1.0000 - 1.0000i \end{bmatrix}$

>> vv=v/v(2,1)  

Scaling to get easier entries.  

$vv = \begin{bmatrix} 0.5000 - 0.5000i \\ 0.5000 + 0.5000i \\ 1.0000 \\ 1.0000 \end{bmatrix}$

Eigen pairs from MATLAB  

$(1+i, \begin{bmatrix} 1/2 - 1/2i \\ 1 \end{bmatrix}), (1-i, \begin{bmatrix} 1/2 + 1/2i \\ 1 \end{bmatrix})$,

$y(t) = C_1 e^{(1+i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} + C_2 e^{(1-i)t} \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

Text answers.  

$\lambda = 1 + i$ and $\overline{\lambda} = 1 - i$.  

$w = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$, $\overline{w} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

Both are correct. WHY?
Complex and real valued solutions

The solutions in form $y(t) = C_1 e^{\lambda t} w + C_2 e^{\lambda t} \overline{w}$ are complex valued. Complex-valued solutions are preferred in some situations (for example, in electrical engineering and physics).

However, in other situations, it is important to find real-valued solutions. Fortunately, the real and imaginary parts of a complex solution provide the needed fundamental set of solutions.

**Theorem** Suppose $A$ is an $n \times n$ matrix with real coefficients, and suppose that $z(t) = x(t) + i \, y(t)$ is a solution to the system $z' = Az$.

(a) The complex conjugate $\overline{z} = x - i y$ is also a solution to $z' = Az$.

(b) The real and imaginary parts $x$ and $y$ are also solutions to $z' = Az$. Furthermore, if $z$ and $\overline{z}$ are linearly independent, so are $x$ and $y$.

Proof: For (a) just form the conjugate of $z' = Az$ and recall that $A$ is a real matrix;

$$\overline{z}' = \overline{A} z \Rightarrow \overline{z}' = \overline{A} \overline{z} = A \overline{z}$$

For (b) use that $z(t) = x(t) + i \, y(t)$ and $\overline{z}(t) = x(t) - i \, y(t)$ are both solutions hence we can form linear combinations to give other solutions. So we have the real valued solutions

$$x = \frac{1}{2} (z + \overline{z}) \quad \text{and} \quad y = \frac{1}{2i} (z - \overline{z})$$

We need to show $x(t)$ and $y(t)$ are linearly independent. The argument is provided in the text.
What do you have for the complex root case of system $z' = Az$?

In the case of our $2 \times 2$ real matrix $A$ with complex eigenvalue $\lambda = \alpha + i\beta$ and associated eigenvector $w = v_1 + iv_2$, we have the complex valued solution $z(t) = e^{\lambda t}w$.

We need to find the real and imaginary parts of $z(t) = e^{\lambda t}w$. Here a little theory helps out. We have $e^{\lambda t} = e^{(\alpha+\beta)t} = e^{\alpha t}e^{\beta it}$.

We can simplify the complex part $e^{\beta it}$ using Euler’s identity: $e^{\beta it} = \cos(\beta t) + i\sin(\beta t)$.

Then we have

\[
z(t) = e^{\lambda t}w = e^{(\alpha+i\beta)t}(v_1 + iv_2) = e^{\alpha t}(\cos \beta t + i \sin \beta t)(v_1 + iv_2) = e^{\alpha t}(\cos \beta t v_1 - \sin \beta t v_2) + i e^{\alpha t}(\sin \beta t v_1 + \cos \beta t v_2)
\]

As we showed previously, since we have a complex solution by taking the real and imaginary parts, we get a fundamental set of solutions to the system $y' = Ay$.

Thus, the general solution is

\[
y(t) = C_1 e^{\alpha t}(\cos \beta t v_1 - \sin \beta t v_2) + C_2 e^{\alpha t}(\sin \beta t v_1 + \cos \beta t v_2)
\]
Example: Find a fundamental set of solutions to $y' = Ay$ where $A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$

$$y(t) = C_1 e^{(1+i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} + C_2 e^{(1-i)t} \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

We take one complex solution and apply Euler's formula to get

$$e^{it} = \cos(t) + i \sin(t)$$

$$z(t) = e^{(1+i)t} \left( \begin{array}{c} 1 \\ 1+i \end{array} \right)$$

$$= e^t \left[ \cos t + i \sin t \right] \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + i \left[ \cos t \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \sin t \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right]$$

$$= e^t \left[ \cos t \left( \begin{array}{c} 1 \\ 1 \end{array} \right) - \sin t \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right] + i \left[ \cos t \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \sin t \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right]$$

$$= e^t \left( \begin{array}{c} \cos t \\ \cos t - \sin t \end{array} \right) + ie^t \left( \begin{array}{c} \sin t \\ \cos t + \sin t \end{array} \right).$$

We take the real and imaginary parts of $z$ to obtain a pair of real valued solutions which are fundamental set of solutions.

$$x(t) = e^t \left( \begin{array}{c} \cos t \\ \cos t - \sin t \end{array} \right) \quad \text{and} \quad y(t) = e^t \left( \begin{array}{c} \sin t \\ \cos t + \sin t \end{array} \right).$$

So the general real valued solution is

$$y(t) = C_1 e^t \begin{bmatrix} \cos(t) \\ \cos(t) - \sin(t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(t) \\ \cos(t) + \sin(t) \end{bmatrix}.$$
Preferred form of the solution in the complex case:

One form of the solution that stresses the eigen information of $\lambda = \alpha \pm \beta$ and corresponding complex eigenvectors $v_1$ and $v_2$ is

$$y(t) = C_1 e^{\alpha t} (\cos \beta t \, v_1 - \sin \beta t \, v_2) + C_2 e^{\alpha t} (\sin \beta t \, v_1 + \cos \beta t \, v_2)$$

This is still a complex valued solution.

It is preferred that we use Euler's formula in the following way:

for eigenpair

$$(\alpha + \beta i, \, v_1) = \begin{pmatrix} \alpha + \beta i \\ a + bi \\ c + di \end{pmatrix}$$

we have

$$e^{(\alpha + \beta i)t} \begin{bmatrix} a + bi \\ c + di \end{bmatrix} = e^{\alpha t} e^{i \beta t} \begin{bmatrix} a + bi \\ c + di \end{bmatrix} = e^{\alpha t} (\cos(\beta t) + i \sin(\beta t)) \begin{bmatrix} a \\ c \end{bmatrix} + i \begin{bmatrix} b \\ d \end{bmatrix}$$

Next multiply terms and collect into real and imaginary parts.

$$e^{(\alpha + \beta i)t} \begin{bmatrix} a + bi \\ c + di \end{bmatrix} = e^{\alpha t} \begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{bmatrix} a \\ c \end{bmatrix} + i e^{\alpha t} \begin{pmatrix} -\sin(\beta t) & -\cos(\beta t) \\ \cos(\beta t) & \sin(\beta t) \end{pmatrix} \begin{bmatrix} a \\ c \end{bmatrix}$$

Finally we take the real and imaginary parts.

$$y(t) = C_1 e^{\alpha t} \begin{bmatrix} a \cos(\beta t) - b \sin(\beta t) \\ c \cos(\beta t) - d \sin(\beta t) \end{bmatrix} + C_2 e^{\alpha t} \begin{bmatrix} a \sin(\beta t) + b \cos(\beta t) \\ c \sin(\beta t) + d \cos(\beta t) \end{bmatrix}$$
Let’s use our example to check things out.

**Example:** Find a fundamental set of solutions to \( y' = Ay \) where \( A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \)

\( \lambda = 1 + i \) and \( \bar{\lambda} = 1 - i. \)

We found

\[
\begin{align*}
\lambda &= 1 + i \quad \text{and} \quad \bar{\lambda} = 1 - i, \\
\mathbf{w} &= \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}, \\
\mathbf{\bar{w}} &= \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}, \\
y(t) &= C_1 e^{(1+i)t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} + C_2 e^{(1-i)t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix}
\end{align*}
\]

Start with

\[
e^{(1+i)t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \Rightarrow e^{(\alpha+i\beta)t} \begin{pmatrix} a + bi \\ c + di \end{pmatrix} \Rightarrow \alpha = 1, \beta = 1, a = 1, b = 0, c = 1, d = 1
\]

\( y(t) = C_1 e^{\alpha t} \begin{pmatrix} a \cos(\beta t) - b \sin(\beta t) \\ c \cos(\beta t) - d \sin(\beta t) \end{pmatrix} + C_2 e^{\alpha t} \begin{pmatrix} a \sin(\beta t) + b \cos(\beta t) \\ c \sin(\beta t) + d \cos(\beta t) \end{pmatrix} \)

So

\[
y(t) = C_1 e^t \begin{pmatrix} 1 \cos(t) - 0 \sin(t) \\ 1 \cos(t) - 1 \sin(t) \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \sin(t) + 0 \cos(t) \\ 1 \sin(t) + 1 \cos(t) \end{pmatrix}
\]

\[
= C_1 e^t \begin{pmatrix} \cos(t) \\ \cos(t) - \sin(t) \end{pmatrix} + C_2 e^t \begin{pmatrix} \sin(t) \\ \sin(t) + \cos(t) \end{pmatrix}
\]

which is exactly the expression we worked out in detail previously.
Another Example: Find a fundamental set of solutions to $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} -1 & -2 \\ 4 & 3 \end{bmatrix}$

%Using MATLAB for eigen info

>> A=[-1 -2;4 3]
A =
 -1   -2
 4    3

>> [v,d]=eig(A)
v =
 -0.4082 + 0.4082i  -0.4082 - 0.4082i
 0.8165             0.8165
d =
 1.0000 + 2.0000i  0
 0               1.0000 - 2.0000i

>> vv=v/v(2,1)
vv =
 -0.5000 + 0.5000i  -0.5000 - 0.5000i
 1.0000             1.0000

Let $\lambda = \alpha + i\beta = 1 + 2i$ and

$$\begin{bmatrix} a + bi \\ c + di \end{bmatrix} = \begin{bmatrix} -0.5 + 0.5i \\ 1 \end{bmatrix}$$

Then $\alpha = 1$, $\beta = 2$, $a = -0.5$, $b = 0.5$, $c = 1$, $d = 0$ so using

$$\mathbf{y}(t) = C_1e^{\alpha t} \begin{bmatrix} a\cos(\beta t) - b\sin(\beta t) \\ c\cos(\beta t) - d\sin(\beta t) \end{bmatrix} + C_2e^{\alpha t} \begin{bmatrix} a\sin(\beta t) + b\cos(\beta t) \\ c\sin(\beta t) + d\cos(\beta t) \end{bmatrix}$$

we have

$$\mathbf{y}(t) = C_1e^{t} \begin{bmatrix} -0.5\cos(2t) - 0.5\sin(2t) \\ \cos(2t) \end{bmatrix} + C_2e^{t} \begin{bmatrix} -0.5\sin(2t) + 0.5\cos(2t) \\ \sin(2t) \end{bmatrix}$$

This is exercise #17 in Section 9.2. The textbook answer is given as

$$\begin{align*}
y_1(t) &= e^{t} \begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix}, \\
y_2(t) &= e^{t} \begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}
\end{align*}$$

How could you check each answer?