Section 9.1 Over view of the technique (review of eigen)

Key Terms:

• Eigenvalues and eigenvectors (review)
• Eigenspace
• Computing eigen Information
• Case of distinct eigenvalues
• Fundamental set of solutions
• Solution of $y' = Ay$ when $A$ has distinct real eigenvalues
**Definition** Let $A$ be an $n \times n$ matrix. The scalar $\lambda$ is called an **eigenvalue** of matrix $A$ if there exists an $n \times 1$ vector $v, v \neq 0$, such that

$$Av = \lambda v.$$ 

Every nonzero vector $v$ satisfying this matrix equation is called an **eigenvector of $A$ associated with eigenvalue** $\lambda$.

Equation $Av = \lambda v$ is commonly called the **eigen equation**.

Let $S$ be the set of all eigenvectors associated with an eigenvalue of matrix $A$ together with the zero vector. Then $S$ is a subspace of $\mathbb{R}^n$. $S$ is called the **eigenspace** associated with eigenvalue $\lambda$.

**Proof:** Let $p$ and $q$ be members of $S$. Then $Ap = \lambda p$ and $Aq = \lambda q$ so $A(p + q) = Ap + Aq = \lambda p + \lambda q = \lambda (p + q)$ $\Rightarrow$ so $S$ is closed under addition.

Let $r$ be any nonzero scalar. Then $A(rp) = r(Ap) = r(\lambda p) = \lambda (rp)$ $\Rightarrow$ so $S$ is closed under scalar multiplication.

$S$ is a subspace.

**Observation:** If $\lambda$ is called an eigenvalue of $A$, then equation $Av = \lambda v$ can be rewritten as $Av = \lambda Iv$ which is equivalent to $(A - \lambda I)v = 0$. Thus the eigenvectors corresponding the $\lambda$ are in the **null space** of matrix $A - \lambda I$, which we denote as $\text{ns}(A - \lambda I)$. 
Computing Eigen Information

The eigen equation can be rearranged as follows:

\[ Av = \lambda v \iff Av = \lambda I_n v \iff Av - \lambda I_n v = 0 \iff (A - \lambda I_n)v = 0 \] (1)

The matrix equation \((A - \lambda I_n)v = 0\) is a homogeneous linear system with coefficient matrix \(A - \lambda I_n\). Since an eigenvector \(v\) cannot be the zero vector, this means we seek a nontrivial solution to the linear system \((A - \lambda I_n)v = 0\). Thus \(\text{ns}(A - \lambda I_n) \neq 0\) or equivalently \(\text{rref}(A - \lambda I_n)\) must contain a zero row. It follows that matrix \(A - \lambda I_n\) must be singular,

\[ \det(A - \lambda I_n) = 0. \quad \text{(or } \det(\lambda I_n - A) = 0 \text{) } \] (2)

Equation (2) is called the characteristic equation of matrix \(A\) and solving it for \(\lambda\) gives us the eigenvalues of \(A\). Because the determinant is a linear combination of particular products of entries of the matrix, the characteristic equation is really a polynomial in \(\lambda\) equation of degree \(n\). We call

\[ c(\lambda) = \det(A - \lambda I_n) \] (3)

the characteristic polynomial of matrix \(A\). The eigenvalues are the solutions of (2) or equivalently the roots of the characteristic polynomial (3). Once we have the \(n\) eigenvalues of \(A\), \(\lambda_1, \lambda_2, \ldots, \lambda_n\), the corresponding eigenvectors are nontrivial solutions of the homogeneous linear systems

\[ (A - \lambda_i I_n)v = 0 \quad \text{for } i = 1, 2, \ldots, n. \] (4)
We summarize the computational approach for determining eigenpairs \((\lambda, \mathbf{v})\) as a two-step procedure:

**Step 1.** To find the eigenvalues of \(\mathbf{A}\) compute the roots of the characteristic polynomial \(\det(\mathbf{A} - \lambda \mathbf{I})\).

**Step 2.** To find an eigenvector corresponding to an eigenvalue \(\lambda\) of \(\mathbf{A}\) compute determine a nontrivial solution to the homogeneous linear system \((\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}\).

**Example:** Find eigenpairs of \(\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\).

**Step 1.** Find the eigenvalues.

\[
c(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}_2) = \det\begin{bmatrix} 1 - \lambda & 1 \\ -2 & 4 - \lambda \end{bmatrix} = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6
\]

Thus the characteristic polynomial is a quadratic and the eigenvalues are the solutions of \(\lambda^2 - 5\lambda + 6 = 0\). We factor the quadratic to get \((\lambda - 3)(\lambda - 2) = 0\) so the eigenvalues are \(\lambda_1 = 3\) and \(\lambda_2 = 2\).
The eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \).

**Step 2.** To find corresponding eigenvectors we solve \((A - \lambda_i I_n)x = 0\) for \(i = 1, 2\).

Case \( \lambda_1 = 3 \): We have that \((A - 3I_2)x = 0\) has augmented matrix
\[
\begin{bmatrix}
-2 & 1 & 0 \\
-2 & 1 & 0
\end{bmatrix}
\]
and its ref is
\[
\begin{bmatrix}
1 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
(Verify.) Thus if \(x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\) we have
\[
x_1 = (1/2)x_2 \text{ so } x = \begin{bmatrix} (1/2)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, x_2 \neq 0. \]
Choosing \(x_2 = 2\), to conveniently get integer entries, gives eigenvector \(x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\).

Case \( \lambda_2 = 2 \): We have that \((A - 2I_2)p = 0\) has augmented matrix
\[
\begin{bmatrix}
-1 & 1 & 0 \\
-2 & 2 & 0
\end{bmatrix}
\]
and its ref is
\[
\begin{bmatrix}
1 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
(Verify.) Thus if \(x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\) we have
\[
x_1 = x_2 \text{ so } x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 \neq 0. \]
Choosing \(x_2 = 1\) gives eigenvector 
\[
x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]
Given that $\lambda$ is an eigenvalue of $A$, then we know that matrix $A - \lambda I_n$ is singular and hence $\text{rref}(A - \lambda I_n)$ will have at least one zero row.

A homogeneous linear system whose coefficient matrix has rref with at least one zero row will have a solution set with at least one free variable. The free variables can be chosen to have any value as long as the resulting solution is not the zero vector.

Since there will be at least one free variable when determining an eigenvector to correspond to an eigenvalue, there are infinitely many ways to express the entries of an eigenvector.
If $(\lambda, p)$ and $(\mu, q)$ are eigenpairs of $A$ with $\lambda \neq \mu$, then $p$ and $q$ are linearly independent.

**Proof:** We will use the following: if vectors $p$ and $q$ are not scalar multiples of one another, then they are linearly independent.

Since $(\lambda, p)$ is an eigenpair of $A$ then $A p = \lambda p$. Similarly since $(\mu, q)$ is an eigenpair of $A$ then $A q = \mu q$. Let $s$ and $t$ be scalars so that $t p + s q = 0$, or equivalently $t p = -s q$. Multiplying this linear combination by $A$ we get

$$A( t p + s q) = t(A p) + s(A q) = t(\lambda p) + s(\mu q) = \lambda (t p) + \mu (s q) = 0 .$$

Substituting for $t p$ we get $\lambda (-s q) + \mu (s q) = 0$ or equivalently that $s(\lambda - \mu) q = 0$. Since $q \neq 0$ and $\lambda \neq \mu$, it must be that $s = 0$, but then $t p = -0 q = 0$. It follows that since $p \neq 0$ then $t = 0$. Hence the only way $t p + s q = 0$ is when $t = s = 0$, so $p$ and $q$ are linearly independent.

This property implies that eigenvectors corresponding to distinct eigenvalues of a matrix $A$ are linearly independent.
Using eigen information to solve $y' = Ay$

We turn to the task of finding the general solution to linear, homogeneous equations and systems.

Remember the strategy we developed in the previous chapter. For a system of dimension $n$, we need to find a fundamental set of solutions, which is a set of $n$ linearly independent solutions.

The general solution is a linear combination of these solutions.

We will also find the general solution for higher-order equations. However, this will be an easy job, since we need only use the results for the associated first-order system.
We are looking for solutions to the system \( y' = Ay \) where \( A \) is a matrix with constant entries.

For motivation, let’s look at some systems we already understand. The simplest example is a system of dimension 1. This reduces to a single, first-order, homogeneous equation with constant coefficients and has the form \( y' = ay \).

Previously we saw that the solution to this equation is the exponential function \( y(t) = Ce^{at} \), where \( C \) is any constant.

Since it works in dimension 1 it is reasonable to look for solutions to \( y' = Ay \) that have an exponential character.

In analogy to the solution found in the 1 dimensional case let’s look for solutions of the form \( y(t) = e^{\lambda t}v \), where \( v \) is a vector with constants for entries. The entries of \( v \) and the constant \( \lambda \) are as yet unknown and can be chosen to allow \( y(t) = e^{\lambda t}v \) to be a solution to \( y' = Ay \). So substituting we have

\[
(e^{\lambda t}v)' = \lambda e^{\lambda t}v = A e^{\lambda t}v \Rightarrow e^{\lambda t} (A v) = e^{\lambda t} (\lambda v)
\]

Since \( e^{\lambda t} \neq 0 \), we have \( y' = Ay \), provided that \( Av = \lambda v \).

This is just the eigen equation; thus to solve \( y' = Ay \) we need eigenvalues and eigenvectors of \( A \).
**Example:** Solve $y' = Ay$ where $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

In our previous example we found eigen pairs $\left(3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ and $\left(2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Thus we have two solutions

$$y_1(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad y_2(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These two solutions are linearly independent, since they are linearly independent for $t = 0$. (We developed this idea in Chapter 8.) But also the eigenvalues are “distinct” so the eigenvectors are linearly independent.

Consequently, they form a fundamental set of solutions.

Hence the general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
If $A$ is 2 by 2 with **real and distinct eigenvalues** then the general solution is

$$y(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$$

where $C_1$ and $C_2$ are arbitrary constants and $v_1$ and $v_2$ are eigenvectors associated with $\lambda_1$ and $\lambda_2$, respectively.

This of course generalizes to an $n$ by $n$ matrix with real distinct eigenvalues where the general solution is.

$$y(t) = \sum_{j=1}^{n} C_j e^{\lambda_j t} v_j$$
**General Case:**

For a system of dimension $n$, the characteristic polynomial of matrix $A$ is of degree $n$. In general, a polynomial of degree $n$ has $n$ roots. Each root $\lambda$ is an eigenvalue, and for each we can find an eigenvector $v$.

From these, we can form the exponential solution $y(t) = e^{\lambda t}v$. That’s $n$ solutions. If these are linearly independent we should be through. However, there are some complications that we will look into in the following sections.

1. **Distinct real roots:** We are essentially done here. Since we can find $n$ linearly independent eigenvectors.

2. **Complex roots:** If an eigenvalue is complex, then the exponential solution is complex valued. (Like $e^{(a + bi)t}$.) Since we will usually want real-valued solutions, this is a complication that we will have to deal with.

3. **Repeated roots:** Sometimes the roots of a polynomial are not distinct. If that polynomial is a characteristic polynomial, then there are fewer distinct eigenvalues than $n$. For each eigenvalue, we are guaranteed only one solution. Hence, our method will give us fewer solutions than we are looking for. This complication will also be dealt with in later sections.
**Very troublesome case:**

A matrix $A$ is called **defective** if $A$ has an eigenvalue $\lambda$ of multiplicity $m > 1$ for which the corresponding eigenspace has a basis of **fewer** than $m$ vectors; that is, the dimension of the eigenspace corresponding to $\lambda$ is less than $m$.

If matrix $A$ in the DE system $y' = Ay$ is defective we will be unable to find a fundamental set of solutions which have **exponential form**.