Section 8.5 Properties of Linear Systems

Key Terms:

• Homogenous Linear systems
  • Linearity property for solutions
  • Linear independence/dependence of solutions
  • Fundamental set of solutions
• Wronskian
Linear Algebra Systems (square)

Ax = 0  $\Leftarrow$ Homog.
Ax = b, b \neq 0  $\Leftarrow$ Nonhomog.

For Ax = 0, if y and z are solutions then so is any linear combination of y and z; this not true for systems Ax = b, b \neq 0. Homogeneous systems have the linearity property.

The null space of Ax = 0 is a subspace and we can find a basis.

If A is nonsingular there is a unique solution.

Linear Differential Systems (square)

x' = Ax  $\Leftarrow$ Homog.
x' = Ax + f, f \neq 0  $\Leftarrow$ Nonhomog.

Suppose that x_1, x_2, \ldots, and x_k are solutions to x' = Ax. Then any linear combination of x_1, x_2, \ldots, and x_k is also a solution; this not true for x' = Ax + f, f \neq 0. Homogeneous systems have the linearity property.

We need the analog of a basis for homogeneous differential systems. That is, we need to determine a set of solutions so that every solution can be written as a linear combination of this special set of solutions.

So we need linearly independent solutions which span the set of all solutions.
Next we have the analog of a basis.

Suppose $y_1, \ldots, \text{and } y_n$ are linearly independent solutions to the $n$-dimensional linear system

$$y'(t) = Ay(t).$$

Then any solution $y$ can be expressed as a linear combination of $y_1, \ldots, \text{and } y_n$. That is, there are constants $C_1, \ldots, \text{and } C_n$ such that

$$y(t) = C_1y_1(t) + \cdots + C_ny_n(t) \text{ for all } t.$$

Thus the general solution to a homogeneous linear system can be expressed as $y = C_1y_1(t) + \cdots + C_ny_n(t)$, when $y_1, \ldots, \text{and } y_n$ are linearly independent.

We say that a set of $n$ linearly independent solutions to a homogeneous, linear system of DEs is a fundamental set of solutions.
Linear Differential Systems continued

Some Theory (No proofs; see P. 365.)

Suppose that $y_1(t)$, $y_2(t)$, . . . , and $y_k(t)$ are solutions to the n-dimensional homogeneous system $y' = Ay$ defined on the interval $I = (\alpha, \beta)$.

1. If we can find a value $t_0$ in $(\alpha, \beta)$ so the set of k n-vectors of constants $y_1(t_0)$, $y_2(t_0)$, . . . , and $y_k(t_0)$ is a linearly dependent set, then the solutions $y_1(t)$, $y_2(t)$, . . . , and $y_k(t)$ are linearly dependent for all values of $t$ in $(\alpha, \beta)$.

2. If there is a value $t_0$ in $(\alpha, \beta)$ so that the set of k n-vectors of constants $y_1(t_0)$, $y_2(t_0)$, . . . , and $y_k(t_0)$ is a linearly independent set then the solutions $y_1(t)$, $y_2(t)$, . . . , and $y_k(t)$ are linearly independent for all values of $t$ in $(\alpha, \beta)$.

We make the following definition.

**Definition:** A set of k solutions that $y_1(t)$, $y_2(t)$, . . . , and $y_k(t)$ to the linear system $y' = Ay$ is linearly independent if it is linearly independent for any one value of $t$ in $(\alpha, \beta)$.

The proofs of the statements above are based on the linearity property and the following Existence and Uniqueness Theorem for Linear Systems of DEs.
We have the following **general result**: (no proof)

Suppose that $A = A(t)$ is an $n \times n$ matrix and $f(t)$ is a column vector and that the components of both are continuous functions of $t$ in an interval $(\alpha, \beta)$. Then, for any $t_0 \in (\alpha, \beta)$, and for any $y_0 \in \mathbb{R}^n$, the inhomogeneous system

$$y'(t) = Ay(t) + f(t),$$

with initial condition $y(t_0) = y_0$, has a **unique solution** defined for all $t \in (\alpha, \beta)$.

This holds if $f(t) = 0$; so it is valid for homogeneous systems.
Solution Strategy for homogeneous system $y' = Ay$

Find $n$ linearly independent solutions $y_1, y_2, \ldots, y_n$. The general solution is the set of all linear combinations,

$$y = C_1y_1 + C_2y_2 + \cdots + C_ny_n,$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Note: if we had IVP $y' = Ay, y(t_0) = y_0$, then to find the particular solution that satisfies the initial condition we need to be able to solve the linear system of algebraic equations generated from the linear combination

$$y_0 = C_1y_1(t_0) + C_2y_2(t_0) + \cdots + C_ny_n(t_0).$$

In matrix form we have

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = y_0$$

Since we have a fundamental set of solutions which are linearly independent the coefficient matrix is nonsingular hence there will be a unique solution to the IVP.
We need a quick way to decide when \( n \) functions are linearly independent. There is a way to do this when the functions are all solutions of the same linear homogeneous system of DEs. It can be shown that we can test to see if a set of solutions \( y_1, y_2, \ldots, y_n \) to \( y' = Ay \) is linearly independent using a determinant computation. According to our development we only have to show that the set of solutions \( y_1, y_2, \ldots, y_n \) is linearly independent for one value of \( t \) in \((\alpha, \beta)\). One way to do this is to find one value of \( t \) so that

\[
W(t) = \det ([y_1(t), y_2(t), \ldots, y_n(t)]) \neq 0.
\]

The function \( W(t) \) is called the **Wronskian of** \( y_1, y_2, \ldots, y_n \).
The idea of the Wronskian needs some elaboration. We start with some linear algebra involving vectors of constants.

Let \( \mathbf{v} = \begin{bmatrix} a \\ c \end{bmatrix} \) and \( \mathbf{w} = \begin{bmatrix} b \\ d \end{bmatrix} \). Then we have the following set of equivalent statements:

The set \( \{\mathbf{v}, \mathbf{w}\} \) is linearly independent if and only if \( C_1 \mathbf{v} + C_2 \mathbf{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) provide \( C_1 = C_2 = 0 \) is the only solution.

Thus augmented matrix \( \begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix} \) has reduced row echelon form \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \) which implies that matrix \( \mathbf{A} = [\mathbf{v} \ \mathbf{w}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is nonsingular and so \( \det(\mathbf{A}) \neq 0 \)

Now consider the DE system \( \frac{dy}{dt} = \mathbf{A} \mathbf{y} \). Suppose that we have two vector functions

\[ Y_1(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \text{ and } Y_2(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \]

which are solutions to the system. (If we can show they are linearly independent then we a fundamental set of solutions.)
Define the Wronskian \( W(t) \) of \( Y_1(t) \) and \( Y_2(t) \) to be the determinant of the matrix whose columns are \( Y_1(t) \) and \( Y_2(t) \). That is,

\[
W(t) = \det \begin{bmatrix} Y_1(t) & Y_2(t) \end{bmatrix} = \det \begin{bmatrix} u_1(t) & p_1(t) \\ u_2(t) & p_2(t) \end{bmatrix} = u_1(t)p_2(t) - u_2(t)p_1(t)
\]

So \( W(t) \) is just a (standard) function.

Next we construct a DE involving \( W(t) \). (For now we omit the dependence on \( t \).)

\[
\frac{dW}{dt} = \frac{d}{dt}(u_1p_2 - u_2p_1) = (u_1p_2' + u_1'p_2) - (u_2p_1' + u_2'p_1)
\]

Since \( Y_1(t) \) and \( Y_2(t) \) are solutions of \( \frac{dy}{dt} = A\mathbf{y} \) we have that

\[
\frac{d}{dt} Y_1(t) = AY_1(t) \Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} au_1 + bu_2 \\ cu_1 + du_2 \end{bmatrix}
\]

\[
\frac{d}{dt} Y_2(t) = AY_2(t) \Rightarrow \begin{bmatrix} p_1' \\ p_2' \end{bmatrix} = \begin{bmatrix} ap_1 + bp_2 \\ cp_1 + dp_2 \end{bmatrix}
\]

Now we replace the derivatives in expression with the components from the expressions.
We get
\[
\frac{dW}{dt} = (u_1 p_2' + u_1' p_2) - (u_2 p_1' + u_2' p_1) = u_1 [c p_1 + d p_2] + [a u_1 + b u_2] p_2 \\
- u_2 [a p_1 + b p_2] - [c u_1 + d u_2] p_1
\]

Expanding and simplifying a number of terms add to zero leaving
\[
\frac{dW}{dt} = (a + d) u_1 p_2 - (a + d) u_2 p_1 = (a + d) W
\]

Here is our DE for \( W \). It is first order linear so the solution is \( W = C e^{(a+d)t} \)

Note this says that function \( W(t) \) is given by \( W(t) = C e^{(a+d)t} \)

Now we get a bit clever:

If we had initial condition \( W(0) = 0 \), then \( C = 0 \) and function \( W(t) \) is identically zero for all values of \( t \).

On the other hand if the initial condition is \( W(0) \neq 0 \), then \( C \neq 0 \) and function \( W(t) \) is a scalar multiple of exponential function \( e^{(a+d)t} \) and so \( W(t) \) is never zero for any value of \( t \).
If $Y_1(0)$ and $Y_2(0)$ are linearly independent columns of constants then (think linear algebra)

$$W(0) = \det \begin{bmatrix} Y_1(0) & Y_2(0) \end{bmatrix} = \det \begin{bmatrix} u_1(0) & p_1(0) \\ u_2(0) & p_2(0) \end{bmatrix} = u_1(0)p_2(0) - u_2(0)p_1(0) \neq 0$$

So by our previous argument the solution of the DE involving $W(t)$ is such that

$$W(t) = Ce^{(a+d)t}$$

where $C \neq 0$ and this implies that

$$W(t) = \det \begin{bmatrix} Y_1(t) & Y_2(t) \end{bmatrix} \neq 0$$

for all $t$. Then it must be that the columns $Y_1(t)$ and $Y_2(t)$ are linearly independent for all $t$, hence they form a fundamental set.

This generalizes so we need only have to show that if the set of solutions $y_1, y_2, \ldots, y_n$ to the system $y' = Ay$ is linearly independent for one value of $t$ in $(\alpha, \beta)$. One way to do this is to find one value of $t$ so that the Wronskian has the property

$$W(t) = \det ([y_1(t), y_2(t), \ldots, y_n(t)]) \neq 0.$$
**Example:** The linear system \( \mathbf{x}'(t) = A\mathbf{x}(t) = \begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix} \mathbf{x}(t) \) has solutions

\[
\mathbf{x}_1(t) = \begin{bmatrix} e^{4t}\cos(2t) \\ e^{4t}\sin(2t) \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} -e^{4t}\sin(2t) \\ e^{4t}\cos(2t) \end{bmatrix}
\]

(a) Show the \( \mathbf{x}_1(t) \) and \( \mathbf{x}_2(t) \) are a fundamental set of solutions.

(b) Determine the particular solution when \( \mathbf{x}(0) = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \).

For (a) we have that \( \mathbf{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \mathbf{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) so our theory implies since these vectors are linearly independent, so are the functions \( \mathbf{x}_1(t) \) and \( \mathbf{x}_2(t) \). Since the system is 2-dimensional it follows that solutions \( \mathbf{x}_1(t) \) and \( \mathbf{x}_2(t) \) are a fundamental set.

For (b) let \( \mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \). Set \( t = 0 \) and we have

\[
\mathbf{x}(0) = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}
\]

Thus \( C_1 = 5 \) and \( C_2 = -3 \) so the solution of the IVP is

\[
\mathbf{x}_1(t) = 5\begin{bmatrix} e^{4t}\cos(2t) \\ e^{4t}\sin(2t) \end{bmatrix} - 3\begin{bmatrix} -e^{4t}\sin(2t) \\ e^{4t}\cos(2t) \end{bmatrix}
\]

This was a very easy example.
**Example:** The linear system $x'(t) = Ax(t) = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} x(t)$ has solutions

$$x_1(t) = \begin{bmatrix} e^{9t} \\ e^{6t} \\ e^{2t} \end{bmatrix}, x_2(t) = \begin{bmatrix} e^{9t} \\ -2e^{6t} \\ 0 \end{bmatrix}, \text{ and } x_3(t) = \begin{bmatrix} e^{9t} \\ e^{6t} \\ -e^{2t} \end{bmatrix}$$

(a) Show the $x_1(t)$, $x_2(t)$, and $x_3(t)$ are a fundamental set of solutions.

(b) Determine the particular solution when $x(0) = \begin{bmatrix} 4 \\ 13 \\ -3 \end{bmatrix}$.

For (a) we have that $x_1(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2(0) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, and $x_3(0) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. The matrix consisting of these 3 columns has determinant of 6 hence the matrix is nonsingular and its columns are linearly independent. So are the functions $x_1(t)$, $x_2(t)$, and $x_3(t)$ are linearly independent. Since the system is 3-dimensional it follows that solutions $x_1(t)$, $x_2(t)$, and $x_3(t)$ are a fundamental set.

For (b) let $x(t) = C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t)$. Set $t = 0$ and we have linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ -3 \end{bmatrix}$$

which has solution $\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$. So the solution of the IVP is $x(t) = 2x_1(t) - 3x_2(t) + 5x_3(t)$.