Section 8.4 Linear Systems

Key Terms:

• Linear systems
  • Homogeneous
  • Nonhomogeneous or inhomogeneous; forcing term
• Standard Form
• Matrix Notation
• Two tank mixing example
• Coupled mass spring example
We say that the unknown functions in a system **appear linearly** if there are no products, powers, or higher-order functions involving the unknowns. A system where the unknown functions appear linearly is called a **linear system**.

A **linear system** of differential equations is any set of differential equations having the following form:

\[
\begin{align*}
    x'_1(t) &= a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\
    x'_2(t) &= a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\
    &\vdots \\
    x'_n(t) &= a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t),
\end{align*}
\]

where \(x_1, \ldots, x_n\) are the unknown functions. The **coefficients** \(a_{ij}(t)\) and \(f_i(t)\) are known functions of the independent variable, \(t\), all defined for \(t \in I\), where \(I = (a, b)\) is an interval in \(\mathbb{R}\).

If all of the \(f_i(t) = 0\), the system is said to be **homogeneous**. Otherwise it is **inhomogeneous** or **nonhomogeneous**.

The functions \(f_1, \ldots, f_n\) are therefore called the **inhomogeneous parts**. The inhomogeneous part is sometimes called the **forcing term**, since in physical systems inhomogeneous parts usually arise as external forces.
Matrix notation for linear systems

Of course we write this in terms of vector functions and matrices as

\[
\begin{pmatrix}
    x'_1(t) \\
    x'_2(t) \\
    \vdots \\
    x'_n(t)
\end{pmatrix} =
\begin{pmatrix}
    a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\
    a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)
\end{pmatrix}
\begin{pmatrix}
    x_1(t) \\
    x_2(t) \\
    \vdots \\
    x_n(t)
\end{pmatrix} +
\begin{pmatrix}
    f_1(t) \\
    f_2(t) \\
    \vdots \\
    f_n(t)
\end{pmatrix}
\]

Matrix Product  Inhomogeneous Part

where \( A = A(t) \) is the \( n \times n \) matrix of the coefficients \( a_{i,j} = a_{i,j}(t) \), \( x = x(t) = (x_1(t), \ldots, x_n(t))^T \), and \( f = f(t) = (f_1(t), \ldots, f_n(t))^T \). We will frequently suppress explicit reference to the independent variable and write it as

\[
x'(t) = A(t)x(t) + f(t)
\]

\( x' = Ax + f \).
Mixing problems, systems

Consider the two tanks in the figure. We have two tanks, connected by two pipes. Each tank contains 500 gallons of a salt solution. Through one pipe solution is pumped from the first tank to the second at 1 gal/min. Through the other, solution is pumped at the same rate from the second tank to the first. Construct the appropriate systems of DEs.

Let $x_1(t)$ and $x_2(t)$ represent the salt content in the two tanks, measured in pounds.

$$\frac{dx_1}{dt} = \text{Rate in} - \text{Rate out} = -\frac{x_1}{500} + \frac{x_2}{500}$$

$$\frac{dx_2}{dt} = \text{Rate in} - \text{Rate out} = \frac{x_1}{500} - \frac{x_2}{500}$$

We have the homogeneous linear system

$$x' = Ax, \quad \text{where} \quad A = \begin{pmatrix} -1/500 & 1/500 \\ 1/500 & -1/500 \end{pmatrix}.$$
Now let’s give some initial conditions which will be initial amounts of salt in each tank.

\[ x_1(0) = 5 \text{ lbs, } x_2(0) = 2 \text{ lbs} \]

It can be shown that the solution of the system is given by

\[
x_1(t) = \exp(-t/250) \ast \left( \frac{7\ast \exp(t/250)}{2} \right) + \frac{3}{2}
\]

\[
x_2(t) = \exp(-t/250) \ast \left( \frac{7\ast \exp(t/250)}{2} \right) - \frac{3}{2}
\]

What is the amount of salt in each tank after a very long time?
Coupled Mass Spring System

Consider a system of two mass, $m_1$ and $m_2$, connected by two springs. One spring is attached to a wall and the other is attached to each of the masses. Assume that the masses slide horizontally on a frictionless surface and there are no external forces driving the system.

There are forces due to the springs that act on the masses. Hooke’s law of springs says for small displacements that the force due to the spring is proportional to the length of the spring displaced from its equilibrium position. This force is expressed as $-kx$ where $x$ is the position of mass from its equilibrium position when the system is set in motion. Here $k$ is considered positive and represents a measure of stiffness of the spring. The minus sign reflects that the fact that the force always acts to restore the spring to its natural equilibrium position.

Let $x(t)$ and $y(t)$ denote the displacements of mass $m_1$ and mass $m_2$, respectively, from the equilibrium positions. We will assume that $x(t)$ and $y(t)$ are positive to the right of their respective equilibrium positions. The stiffness in the springs is denoted by $k_1$ and $k_2$ respectively.
We use Newton's second law, $F = ma$, to construct a DE for each of the masses.

There are two forces acting on mass $m_1$ because each spring is attached. For the first spring there is a force $-k_1x$ (acts toward the left) and for the second spring $+k_2(y - x)$ (which acts toward the right). Since $y$ is greater than $x$, $y - x > 0$, then the second spring is stretched and the force it exerts on the first mass is directed to the right.

So we have

$$m_1 \frac{d^2 x}{dt^2} = -k_1x + k_2(y - x)$$

The second spring exerts the force $-k_2(y - x)$ (the opposite of what the second spring exerts on the first mass).

So we have

$$m_2 \frac{d^2 y}{dt^2} = -k_2(y - x)$$

The system of DEs is

$$m_1 x'' = -k_1x + k_2(y - x)$$

$$m_2 y'' = -k_2(y - x)$$
Next we write the second order system in matrix form as a system of first order DES.

Let \( u_1(t) = x(t) \), \( u_2(t) = x'(t) \), \( u_3(t) = y(t) \), and \( u_4(t) = y'(t) \). Then

\[
\begin{align*}
\dot{u}_2 &= x'' = \frac{- (k_1 + k_2) u_1 + k_2 u_3}{m_1} + \frac{k_2 u_3}{m_1} \\
\dot{u}_4 &= y'' = \frac{k_2 u_1 - k_2 u_3}{m_2} - \frac{k_2 u_3}{m_2}
\end{align*}
\]

Together with the equations \( \dot{u}_1 = x' = u_2 \) and \( \dot{u}_3 = y' = u_4 \), we obtain the system

\[
\begin{align*}
\dot{u}_1 &= u_2 \\
\dot{u}_2 &= x'' = \frac{- (k_1 + k_2) u_1 + k_2 u_3}{m_1} + \frac{k_2 u_3}{m_1} \\
\dot{u}_3 &= u_4 \\
\dot{u}_4 &= y'' = \frac{k_2 u_1 - k_2 u_3}{m_2} - \frac{k_2 u_3}{m_2}
\end{align*}
\]

and in matrix form

\[
\begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\dot{u}_3 \\
\dot{u}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{- (k_1 + k_2)}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\
0 & \frac{1}{m_1} & 0 & \frac{1}{m_1} \\
\frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}
\]