Section 8.2 Geometric Interpretation of Solutions

Key Terms:

• Predator-Prey Example
• Parametric Plot
• Phase Plane
  o Phase Plane Plot or Solution curve
• Secant Vector
• Vector Field
• Direction Field
The objective is to develop a way to interpret a solution to a system of DEs similar to our direction field or slope field for first order DEs. The interpretation will be graphical.

We start with an example called a predator-prey system.

Consider two species that exist together and interact. We will suppose that one of these species is a predator, which depends in an essential way on the other, the prey, for its food supply, and therefore for its existence.

For example foxes & rabbits, wolves & deer, or orcas & seals.

\[
F(t) = \text{prey population (think FOOD!)}
\]
\[
S(t) = \text{predator population}
\]

Each of these we have a reproductive rate; \(r_F\) for the prey and \(r_S\) for the predator. So for rates of change in the populations we have

\[
F' = r_F F
\]
\[
S' = r_S S.
\]
It is the modeler’s task to figure out what the reproductive rates are.

Let’s start with $r_F$, the reproductive rate for the prey population. Assume that there are sufficient food resources so that in the absence of predators, the prey population would follow the Malthusian model with a positive reproductive rate. That means

$$r_F = a > 0, \text{ if } S = 0.$$ 

Now we have to modify the prey reproduction rate to take predation into account.

If there are predators, then $S > 0$ and each encounter of a prey with a predator has a certain probability of resulting in the capture and death of the prey. So the reproductive prey rate will decrease because of an increasing death rate resulting from predation. We will assume that the decrease would be proportional to the number of encounters with predators. Assuming random distribution of prey and predator the number of encounters for an individual prey would be proportional to $S$, the number of predators. Hence the decrease in the reproductive rate would also be proportional to $S$. Therefore,

$$r_F = a - bS, \quad a, b > 0,$$

for some constants $a$ and $b$. 
The analysis of $r_S$, the reproductive rate for the predators, proceeds similarly. In the absence of prey, the predators cannot reproduce well so

$$r_S = -c < 0 \text{ if } F = 0.$$  

The presence of the prey would increase the reproductive rate, and as before, this increase would be proportional to the size of the prey population. So an adjusted predator reproduction rate is

$$r_S = -c + dF \quad c, d > 0.$$  

So the predator-prey DE system is

$$F' = (a - bS)F,$$
$$S' = (-c + dF)S.$$  

This is the Volterra model of predator–prey populations. The system is nonlinear and autonomous.

Let’s look at a special case where we choose parameters and initial conditions and solve it using a DE solver. (We will use a 4th order R-K method.)

$$F' = (0.4 - 0.01S)F,$$
$$S' = (-0.3 + 0.005F)S,$$
$$F(0) = 40 \text{ and } S(0) = 20$$
\[ F' = (0.4 - 0.01S)F \]
\[ S' = (-0.3 + 0.005F)S \]
\[ F(0) = 40 \text{ and } S(0) = 20 \]

over \([0, 80]\) with \(h = 0.05\)

Both the prey and predator populations appear to be periodic.

Next we construct a parametric plot; ordered pairs \((F(t), S(t))\).
A **parametric plot** is a set of ordered pairs \((F(t), S(t))\). This is also said to be a view of the solution in the **phase plane**. **Now the horizontal axis is prey and the vertical axis is predator.**

Note that the curve is closed, so that it tracks over itself as \(t\) increases. This reinforces our speculation that the solution is periodic.

The “time plot” of prey and predator provided information to generate the parametric plot. **But there is no way to use the phase plane plot to get time information.**
If we want to discover the direction of the curve in the phase plane as $t$ increases at a point $u_0 = u(t_0)$ on the solution curve, we can choose $h > 0$, set $u_1 = u(t_0 + h)$, and look at the vector $u_1 - u_0$. Since it connects the points $u_0$ and $u_1$ on the curve, it is called a secant vector. (Naturally we need to plot $u_0$ and $u_1$.)

To generate this picture I went to the R-K output. I chose $t_0 = 6$ and found $F= 121$, $S = 34$; then I chose $h = 1$ and found $F= 120$, $S = 47$ (Numbers rounded.) Plotted the asterisks and drew the arrow from $u_0$ to $u_1$. So the closed curve is traced counterclockwise.

If we computed the limit as $h \to 0$ of

$$
\frac{1}{h}(u(t_0 + h) - u(t_0)) = \frac{1}{h} \left( \frac{F(t_0 + h) - F(t_0)}{S(t_0 + h) - S(t_0)} \right)
$$

we get tangent vector

$$
u'(t_0) = \begin{pmatrix} F'(t_0) \\ S'(t_0) \end{pmatrix}$$
The analysis that we have just done for curves in the plane also works for curves in higher dimension. In particular, if $t \rightarrow y(t)$ is the parameterization for a curve in $\mathbb{R}^n$, then $y'(t)$ is a vector that is tangent to the curve at the point $y(t)$. Unfortunately, the ease of visualization is gone, especially if $n$ is larger than 3. As is often the case, the best we can do for curves in higher dimension is to use the two-dimensional analogy.
The phase space and the phase plane (formal definitions, etc.)

Suppose in general that we have a planar system
\[
\begin{align*}
y'_1 &= f(t, y_1, y_2), \\
y'_2 &= g(t, y_1, y_2).
\end{align*}
\]
If \( y(t) = (y_1(t), y_2(t))^T \) is a solution, then we can look at the solution curve \( t \rightarrow y(t) \) in the plane. In this case, the \( y_1y_2 \)-plane is called the \textbf{phase plane}, and the solution curve is called a \textbf{phase plane plot}, or a solution curve in the phase plane.

For a general system of dimension \( n \), \( x' = f(t, x) \), the space consisting only of the \( x \)-coordinates is called the \textbf{phase space}.

The phase plane is the two-dimensional version of phase space, just as the phase line is the one-dimensional version. A plot of a curve \( t \rightarrow x(t) \) is called a \textbf{phase space plot}. Phase space plots are especially useful in dimension 2 and sometimes in dimension 3. They become even more important for autonomous systems, but they are occasionally used for nonautonomous systems as well.
For a planar autonomous system

\[ \begin{align*}
    x' &= f(x, y), \\
    y' &= g(x, y)
\end{align*} \]

if \( f \) and \( g \) are defined in a rectangle \( R \) in the \( xy \)-plane and we have solution \( (x(t), y(t)) \) then we can determine the corresponding curve \( t \rightarrow (x(t), y(t)) \) in the \( xy \)-plane. At each point \( (x(t), y(t)) \) on the solution curve, the vector

\[ (x'(t), y'(t)) = (f(x(t), y(t)), g(x(t), y(t))) \]

is tangent to the curve.

The set of all such vectors in \( R \) is called the vector field of the system. Notice that the vector field can be computed from the system itself. We do not need to know solutions to find the vector field. (Recall direction fields for first order DEs.)

The vector field for our predator-prey example has vectors

\[ ((0.4 - 0.01S)F, (-0.3 + 0.005F)S). \]

Choosing values for \( F \) and \( S \) gives a vector with initial point \((F, S)\) in the vector field.
Using **pplane8** in MATLAB we have the vector field for the predator-prey example as shown next.

\[
\begin{align*}
x' &= (0.4 - 0.01 y) x \\
y' &= (-0.3 + 0.005 x) y
\end{align*}
\]
One of the options in **pplane8** Solutions drop down menu is to find a nearly closed orbit. A mouse click initiates a starting point. Several are displayed below.

\[ x' = (0.4 - 0.01 y) \, x \]
\[ y' = (-0.3 + 0.005 \, x) \, y \]
In Section 8.1 we used the SIR model. Using the planar autonomous system
\[
S' = -SI,
I' = SI - I,
\]
`pplane8` gives

![Graph showing the solution curve for S(0)=4, I(0)=0.1](graph.png)

infected, I(t)
susceptible, S(t)