Section 2.5 Mixing Problems

Key Terms:

• Tanks
• Mixing problems
• Input rate
• Output rate
• Volume rates
• Concentration
The problems we will discuss are called mixing problems. They employ tanks and other receptacles that hold solutions, mixtures usually containing water and an additional element such as salt. While these examples might appear to be inane, they should not be underestimated. They take on an urgency when the tanks are lakes or body organs.

Consider a lake with a factory on its shore that introduces a pollutant into the lake. The lake is fed by one river and drained by another, keeping the volume of the lake constant. Using the methods we discuss in this section, we can model how the amount of pollutant in the lake varies with time. We can then make intelligent decisions about the danger involved in this situation.

The basic model involves the rate of change of a substance in a “tank” that contains a fluid based on the input of the substance and the output of the substance.

We have

\[
\text{rate in} = \text{volume rate in} \times \text{concentration within the fluid entering}
\]

\[
\text{rate out} = \text{volume rate out} \times \text{concentration within the fluid exiting}
\]

**Units of measurement:**

- Volume rate = gal/min, liters/hr, etc. (time units can vary)
- Concentration = lbs/gal, kg/liter, etc.
Using units we have

\[
\text{rates in or rates out} = \text{volume rate} \times \text{concentration} \\
= \text{gal/min} \times \text{lbs/gal} = \text{lbs/min} \\
\text{or} = \text{liter/hr} \times \text{kg/liter} = \text{kg/hr} \quad \text{ETC.}
\]

So for example the fundamental differential equation is

rate of change of substance = rate in − rate out = lbs/min of input − lbs/min of output

There are three basic cases:

• volume of fluid in the tank remains constant ⇒ volume rate in = volume rate out

• volume of fluid in the tank increases ⇒ volume rate in > volume rate out

• volume of fluid in the tank decreases ⇒ volume rate in < volume rate out
Example:

At time \( t = 0 \) a tank contains \( x_0 \) lb of salt dissolved in 100 gal of water. Assume that water containing \( 1/4 \) lb of salt/gal is entering the tank at a rate of \( r \) gal/min and that the well-stirred mixture is draining from the tank at the same rate.

(a) Construct the IVP that describes \( x(t) \) the amount of salt in the tank at any time \( t \) and find the amount of salt in the tank at any time.

(b) Find the limiting amount \( x_L \), that is present after a very long time.

\[
\text{rate in} = \text{volume rate in} \times \text{concentration within the fluid entering} = r \text{ gal/min} \times \frac{1}{4} \text{ lb/gal} = \frac{r}{4} \text{ lb/min}
\]

\[
\text{rate out} = \text{volume rate out} \times \text{concentration within the fluid exiting} = r \text{ gal/min} \times \frac{x(t)}{100} \text{ lb/gal} = \frac{r x(t)}{100} \text{ lb/min}
\]

IVP \[
\frac{dx}{dt} = \frac{r}{4} - \frac{r x}{100} , \ x(0) = x_0
\]
Solve the IVP: The DE \( \frac{dx}{dt} = \frac{r}{4} - \frac{rx}{100} \) is both separable and first order linear.

Converting to standard form we have \( \frac{dx}{dt} + \frac{rx}{100} = \frac{r}{4} \) so the integrating factor is

\[
u(t) = e^{\int \frac{r}{100} dt} = e^{\frac{rt}{100}}
\]

Multiplying both sides by the integrating factor we have the DE represented by

\[
\frac{d}{dt} \left( xe^{\frac{rt}{100}} \right) = \frac{r}{4} e^{\frac{rt}{100}}
\]

Integrating we get

\[
x e^{\frac{rt}{100}} = \int \frac{r}{4} e^{\frac{rt}{100}} dt = \frac{r}{4} \frac{100}{r} e^{\frac{rt}{100}} + C
\]

Solving for \( x \) we get

\[
x = \frac{r}{4} \frac{100}{r} e^{\frac{-rt}{100}} + Ce^{\frac{rt}{100}}
\]

Applying \( x(0) = x_0 \) we find \( C = x_0 - 25 \)

The solution of the IVP is

\[
x = 25(x_0 - 25)e^{\frac{-rt}{100}}
\]

(b) Find the limiting amount \( x_L \), that is present after a very long time.

\[
x_L = \lim_{t \to \infty} \left( 25 + (x_0 - 25)e^{\frac{-rt}{100}} \right) = 25
\]
\[ x_L = \lim_{t \to \infty} \left( 25 + (x_0 - 25)e^{-rt} \right) = 25 \]

What were the roles of parameters \( r \) and \( x_0 \) in computing \( x_L \)?

Let's look at the solution closely: \[ x = 25 + (x_0 - 25)e^{-rt} \]

\[ x = 25(1 - e^{-100}) + x_0e^{-100} \]

The amount of salt in the tank due to the flow processes.

Portion of the original salt that remains in the tank at time \( t \).
Let’s modify things a bit: set $r = 3$ gal/min and $x_0 = 50$ lbs.

In this case the solution of the IVP is

$$x = 25 + (50 - 25)e^{\frac{-3t}{100}}$$

Find the time $T$ at which $x(t)$ is within 2% of $x_L = 25$. (recall that $x(t) > 25$)

2% of 25 is 0.5 so we want $T$ when $x = 25.5 \Rightarrow 25.5 = 25 + 25e^{\frac{-0.03t}{100}}$

Solve for $t$.

$$\frac{0.5}{25} = e^{\frac{-0.03t}{100}} \Rightarrow \ln\left(\frac{0.5}{25}\right) = -0.03t \Rightarrow T = t = \frac{\ln\left(\frac{0.5}{25}\right)}{-0.03} \approx 130.4$ \text{min.}$$

Next let’s find the rate $r$ so that the time $T$ when $x(t)$ is within 2% of 25 lbs does not exceed 45 min.

Here we use $x = 25 + 25e^{\frac{-rt}{100}}$ We set $t = 45$, $x = 25.5$ and solve for $r$.

We have

$$25.5 = 25 + 25e^{\frac{-rt}{100}}$$

$$\frac{0.5}{25} = e^{\frac{-rt}{100}} \Rightarrow \ln\left(\frac{0.5}{25}\right) = -r(0.45) \Rightarrow r = \frac{\ln\left(\frac{0.5}{25}\right)}{-0.45} \approx 8.69 \text{ gal/min}$$
Example:
A 1000 gallon tank contains 400 gallons of pure water. A valve is opened so that fluid containing 2lbs of salt per gallon enters the tank at the rate of 4 gallons per minute and at the same time a drain valve is opened so fluid exits the tank at 2 gallons per minute. Construct an IVP for this situation.

\[
\frac{dx}{dt} = \text{rate in of substance - rate out of substance}
\]

\[
= (4 \text{ gal/min})(2\text{lbs/gal}) - (2\text{gal/min})\frac{x(t)}{400+2t}\text{lbs/gal}
\]

\[
\frac{dx}{dt} = (4 \text{ gal/min})(2\text{lbs/gal}) - (2\text{gal/min})\frac{x(t)}{400+2t}\text{lbs/gal}, \ x(0) = 0
\]

As time goes on what happens to the tank in this case?

How do you find the time that this occurs? \(t = 300 \text{ min}\)

How much salt is in the tank then? \(x = 1680 \text{ lbs}\)

Solution:

\[
x = \frac{4t(t + 400)}{t + 200}
\]
**Example:**

Consider two tanks, labeled tank A and tank B. Tank A contains 100 gal of solution in which is dissolved 20 lb of salt. Tank B contains 200 gal of solution in which is dissolved 40 lb of salt.

Pure water flows into tank A at a rate of 5 gal/sec. There is a drain at the bottom of tank A. Solution leaves tank A via this drain at a rate of 5 gal/sec and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B, also at a rate of 5 gal/sec.

What is the salt content in tank B after 1 minute?

**Observations:**
- The volume of solution in both tanks remains the same.
- Tank A has pure water entering \( \rightarrow \) input rate of salt = 0
- Tank B has input rate of salt = output rate of salt from tank A

**IVP for tank A:**

\[
\frac{dx}{dt} = 0 - \frac{x}{100} \text{ lbs/gal} \times 5 \text{ gal/sec} = - \frac{1}{20} x
\]

\( x(0) = 20 \text{ lbs} \)

**IVP for tank B:**

\[
\frac{dy}{dt} = \frac{1}{20} x - \frac{y}{200} \text{ lbs/gal} \times 5 \text{ gal/sec} = \frac{1}{20} x - \frac{1}{40} y
\]

\( y(0) = 40 \)
So we have a system of DEs with initial conditions
\[
\begin{align*}
\frac{dx}{dt} &= -\frac{1}{20} x \quad x(0) = 20 \\
\frac{dy}{dt} &= \frac{1}{20} x - \frac{1}{40} y \quad y(0) = 40
\end{align*}
\]

In this case the form of the system allows us to solve the IVP for tank A and then use that solution in the second DE.

Solving the first IVP:
\[
x(t) = C_1 e^{-t/20}
\]
Using \(x(0) = 20\) gives \(C_1 = 20\)
\[
x(t) = 20e^{-t/20}
\]

Solving the second IVP:
\[
\frac{dy}{dt} = \frac{1}{20} (20e^{-t/20}) - \frac{1}{40} y = e^{-t/20} - \frac{1}{40} y
\]
The DE is first order linear
\[
\frac{dy}{dt} + \frac{1}{40} y = e^{-t/20}
\]
Integrating factor \(u(t) = e^{t/40}\)
\[
\frac{d}{dt} \left( e^{t/40} y \right) = e^{t/40} e^{-t/20} = e^{-t/40}
\]
\[
e^{t/40} y = -40e^{-t/40} + C_2 \quad \Rightarrow \quad y = -40e^{-t/20} + C_2 e^{-t/40}
\]
Use \(y(0) = 40\) and \(C_2 = 80\).
\[
y = -40e^{-t/20} + 80e^{-t/40}
\]
Now set \(t = 60\); \(y(60) \approx 15.9\) lbs