Section 3.7 Mass-Spring Systems (no damping)

Key Terms/ Ideas:

• Hooke’s Law of Springs

• Undamped Free Vibrations (Simple Harmonic Motion; SHM also called Simple Harmonic Oscillator)
  - Amplitude
  - Natural Frequency
  - Period
  - Phase Shift

Warning: set your calculator for trig functions to radians NOT degrees.
Here are some basic units in metric and English form. There are times when we need to convert units to their proper form based on the units for acceleration of gravity (denoted by \( g \)).

**English \( g = 32 \text{ ft/sec}^2 \)  \hspace{1cm} **Metric \( g = 9.8 \text{ m/sec}^2 \)

### Basic Mechanical Units

<table>
<thead>
<tr>
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<th>SI Units (MKS)</th>
<th>(CGS)</th>
<th>U.S. Common</th>
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<tr>
<td><strong>Length (L)</strong></td>
<td>meter (m)</td>
<td>centimeter (cm)</td>
<td>foot (ft)</td>
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<tr>
<td><strong>Time (T)</strong></td>
<td>second (s)</td>
<td>second (s)</td>
<td>second (s)</td>
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<tr>
<td><strong>Mass (M)</strong></td>
<td>kilogram (kg)</td>
<td>gram (gm)</td>
<td>slug</td>
</tr>
<tr>
<td><strong>Velocity (L/T)</strong></td>
<td>( \text{m/sec} )</td>
<td>( \text{cm/sec} )</td>
<td>( \text{ft/sec} )</td>
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<tr>
<td><strong>Acceleration (L/T^2)</strong></td>
<td>( \text{m/sec}^2 )</td>
<td>( \text{cm/sec}^2 )</td>
<td>( \text{ft/sec}^2 )</td>
</tr>
<tr>
<td><strong>Force (ML/T^2)</strong></td>
<td>kg ( \text{m/sec}^2 ) = Newton (N)</td>
<td>gm ( \text{cm/sec}^2 ) = dyne</td>
<td>slug ( \text{ft/sec}^2 ) = pound (lb)</td>
</tr>
<tr>
<td><strong>Work (ML^2/T^2)</strong></td>
<td>( N \text{m} = \text{joule (J)} )</td>
<td>dyne cm = erg</td>
<td>lb ft = ft lb</td>
</tr>
<tr>
<td><strong>Energy (ML^2/T^2)</strong></td>
<td>joule</td>
<td>erg</td>
<td>ft lb</td>
</tr>
<tr>
<td><strong>Power (ML^2/T^3)</strong></td>
<td>( j/s = \text{watt (W)} )</td>
<td>erg/s</td>
<td>ft lb/s</td>
</tr>
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slug = 1 pound of force * s^2/ft = lbf * s^2/ft

Viscous damping: metric units N/(m/s)  \hspace{1cm} \text{English units lb/(ft/sec)}
Simple model for Mass-Spring Systems.

We will use this as our generic form of the mass spring system.

Figures adapted from the work of Dr. Tai-Ran Hsu at SJSU and Wikipedia.
We will study the motion of a mass on a spring in detail because an understanding of the behavior of this simple system is the first step in the investigation of more complex vibrating systems.

Natural length of the spring with no load attached.
We attach a body of mass \( m \), and weight \( mg \), to the spring.

The spring is stretched an additional \( L \) units.

The body will remain at rest in a position such that the length of the spring is \( l + L \).

**Equilibrium position** or rest position of the spring-mass system.

http://www.lon-capa.org/~mmp/kap13/cd361a.htm

We take the **downward direction to be positive**.

Next we appeal to Newton’s law of motion: sum of forces = mass times acceleration to establish an IVP for the motion of the system; \( F = ma \).

There are **two forces** acting at the point where the mass is attached to the spring. The **gravitational force**, or **weight** of the mass \( m \) acts **downward** and has magnitude \( mg \), where \( g \) is the acceleration of gravity. There is also a force \( F_s \) **due to the spring**, that acts **upward**.

\[
F = mg + F_s
\]
Let $u(t)$, measured positively downward, denote the displacement of the mass from its equilibrium position at time $t$. We have from Newton’s second law that $F = ma$ and so we are led to the DE

$$mu(t)'' = mg + F_s$$

acceleration of the mass

Equilibrium position.

To determine the force due the spring we use Hooke’s Law.

Hooke’s law of springs says for small displacements that force $F_s$ is proportional to the length of the stretch in the spring. The proportionality constant is a positive value denoted by $k > 0$ so $F_s = -k(L + u(t))$ where $u(t)$ is the position of mass from equilibrium when the system is set in motion. This force always acts to restore the spring to its natural equilibrium position. Since $u$ is function of time it varies in sign as the system oscillates; this force can change direction as $L + u(t)$ changes sign. Regardless of the position of the mass this formula works. Constant $k > 0$ is a measure of stiffness of the spring. Thus we have second order linear DE $mu(t)'' = mg - k(L + u(t)) = mg - kL - k u(t)$.

When at the equilibrium position the two forces must be equal so that $mg = kL$, so this DE can be simplified to the form $mu(t)'' + ku(t) = 0$. (or as $mu'' + ku = 0$)

Computing the spring constant: If a weight $W$ stretches the spring $L$ units at equilibrium, then $k = W/L$. Of course $W = mg$. 

The DE for the motion of the mass is \( mu'' + ku = 0 \).

To get an IVP we specify the auxiliary conditions \( u(0) = u_0, \ u'(0) = v_0 \).

**Text book conventions:** pulling the mass downward implies \( u_0 > 0 \). Releasing the mass from rest implies \( v_0 = 0 \) and giving the mass a push downward implies \( v_0 > 0 \).

**SUMMARY:** The IVP \( mu'' + ku = 0, \ u(0) = u_0, \ u'(0) = v_0 \) is said to model Undamped Free Vibrations (Simple Harmonic Motion) or sometimes the terms Unforced Undamped Oscillations are used. We are assuming that things like air resistance and friction are negligible.

Since \( m \) and \( k \) are positive the roots of the characteristic polynomial are of the form \( \pm \mu i \). The formula for \( u(t) \) is a sinusoid of fixed amplitude.

\[
mu'' + ku = 0, \ \text{characteristic equation} \quad mr^2 + k = 0 \quad \Rightarrow \quad r = \pm i \sqrt{\frac{k}{m}}
\]

\[
u(t) = A \cos \left( \sqrt{\frac{k}{m}} \ t \right) + B \sin \left( \sqrt{\frac{k}{m}} \ t \right)
\]

**General solution of the ODE.**

The graph of \( u(t) \) will be a sinusoid of fixed amplitude. (The picture can vary.)
Mathematical notation and terminology for the case of Simple Harmonic Motion

**IVP:** \( mu'' + ku = 0 \) \hspace{1cm} **General Solution:** \( u(t) = A \cos \left( \sqrt{\frac{k}{m}} t \right) + B \sin \left( \sqrt{\frac{k}{m}} t \right) \)

\( u(0) = u_0, \quad u'(0) = v_0. \)

To ease the notation a bit we **define** \( \omega_0^2 = \frac{k}{m} \) so that the general solution of the DE has the form \( u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) \) and applying the initial conditions \( u(0) = u_0, \quad u'(0) = v_0 \) we can show that \( A = u_0 \) and \( B = \frac{v_0}{\omega_0}. \)

**Natural frequency (or circular frequency) =** \( \omega_0 \) (radians per unit of time; measure of rotation rate)

The function \( u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) \) is often expressed as a multiple of a cosine function with a **shift** in the form \( u(t) = R \cos(\omega_0 t - \delta). \)

To go from \( u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) \) to \( u(t) = R \cos(\omega_0 t - \delta) \) we proceed as follows.

Consider the triangle in the figure. Then we have

\[ R = \sqrt{A^2 + B^2}, \quad \cos(\delta) = \frac{A}{R}, \quad \sin(\delta) = \frac{B}{R} \]

The \( \tan(\delta) = \frac{B}{A} \) and we use the **tangent inverse** to determine the **shift** angle \( \delta. \) But things are not simple. See the following.
Although \( \tan(\delta) = B/A \), the angle \( \delta \) is not given by the principal branch of the inverse tangent function which gives values only in the interval \((-\pi/2, \pi/2)\). Instead \( \delta \) is an angle in the interval between 0 and \( 2\pi \) whose cosine and sine have the same signs given by the expressions for \( \sin(\delta) \) and \( \cos(\delta) \) listed above. In these expressions either \( A \) or \( B \) or both may be negative. Thus we have

\[
\delta = \begin{cases} 
\tan^{-1}(B/A), & \text{if } A > 0, B > 0 \text{ (first quadrant)} \\
\pi + \tan^{-1}(B/A), & \text{if } A < 0 \text{ (second and third quadrant)} \\
2\pi + \tan^{-1}(B/A), & \text{if } A > 0, B < 0 \text{ (fourth quadrant)} 
\end{cases}
\]

where \( \tan^{-1}(B/A) \) is the angle in \((-\pi/2, \pi/2)\) given by computation on a calculator or computer.

In any event we have that \( u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) \) and then

\[
u(t) = R \left( \frac{A}{R} \cos(\omega_0 t) + \frac{B}{R} \sin(\omega_0 t) \right) \\
= R \left( \cos(\delta) \cos(\omega_0 t) + \sin(\delta) \sin(\omega_0 t) \right) \\
= R \cos(\omega_0 t - \delta)
\]

Note the use of a trig. Identity for cosine of a difference of two angles.

We used

\[
R = \sqrt{A^2 + B^2}, \quad \cos(\delta) = \frac{A}{R}, \quad \sin(\delta) = \frac{B}{R}
\]
The graph of \( u(t) = R \cos(\omega_0 t - \delta) \) is a *shifted cosine wave* that describes the *periodic or simple harmonic motion* of the mass at the end of the spring. We have the further information

*Amplitude* = \( R \)

*Natural frequency (or circular frequency)* = \( \omega_0 \) (radians per unit of time; measure of rotation rate)

*Period of motion* = \( T = \frac{2\pi}{\omega_0} = \frac{2\pi}{(k/m)^{1/2}} \) (time for 1 full oscillation)

The dimensionless parameter \( \delta/\omega_0 \) is called the *phase (shift) or phase angle*, and measures the displacement of the wave from its normal corresponding position for \( \delta = 0 \).

The graph of \( u(t) = R \cos(\omega_0 t - \delta) \) can be viewed several ways depending upon how you scale the horizontal axis.

We note that for \( u(t) = R \cos(\omega_0 t - \delta) \) the *maximum magnitude* will occur when \( \omega_0 t - \delta = 0 \) *(Explain!)* which gives \( t = \delta/\omega_0 \). (the first occurrence)

We show two graphs; the first is with the horizontal axis representing \( \omega_0 t \) and the second with the horizontal axis representing \( t \).
$$u(t) = R \cos(\omega_0 t - \delta)$$

Period:

$$\omega_0 t$$

$$\delta$$

$$\delta + 2\pi$$

$$T = \frac{2\pi}{\omega_0}$$

$$\frac{\delta}{\omega_0} = \text{length of}$$
Example:
A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in and then given an initial velocity downward of 4 in/sec. (Assume no damping.) Determine the position $u(t)$ of the mass at any time $t$. Then determine the first time the maximum magnitude will occur.

We have English units so $g = 32 \text{ft/sec}^2$ so change the inches to feet; 6 in = $\frac{1}{2}$ ft, 3 in = $\frac{3}{4}$ ft, 4 in/sec = $\frac{1}{3}$ ft/sec

The IVP: $mu'' + ku = 0$, $u(0) = \frac{1}{4}$ ft, $u'(0) = \frac{1}{3}$ ft/sec

$m = \text{weight/g} = \frac{2}{32} = \frac{1}{16}$ $k = \text{weight/stretch} = 2/\left(\frac{1}{2}\right) = 4$

$(1/16)u'' + 4u = 0 \Rightarrow u'' + 64u = 0$

Characteristic equation: $r^2 + 64 = 0$ so $r = \pm 8i$

So the general solution is $u(t) = A \cos(8t) + B \sin(8t)$.

Applying the initial conditions: $u(0) = \frac{1}{4} \Rightarrow A = \frac{1}{4}$.

$u'(t) = -8A\sin(8t) + 8B\cos(8t)$ then using $u'(0) = \frac{1}{3}$ we get $1/3 = 8B \Rightarrow B = \frac{1}{24}$.

Thus $u(t) = \frac{1}{4} \cos(8t) + \frac{1}{24} \sin(8t)$ Express this in the form $u(t) = R \cos(\omega_0 t - \delta)$

$\omega_0^2 = k/m \Rightarrow \omega_0 = 8$ $R = \sqrt{A^2 + B^2} = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{24}\right)^2} \approx 0.2534$

Both $A$ and $B$ are positive so

$\delta = \tan^{-1}(B/A) = \tan^{-1}(1/6) \approx 0.1651$

$u(t) = 0.2534 \cos(8t - 0.1651)$
\[ mu'' + ku = 0, \quad u(0) = \frac{1}{4} \text{ ft, } u'(0) = 1/3 \text{ ft/sec} \quad (1/16)u'' + 4u = 0 \quad \Rightarrow \quad u'' + 64u = 0 \]

\[ u(t) = 0.2534 \cos(8t - 0.1651) \]

The maximum magnitude will occur when \( 8t - 0.1651 = 0 \) which gives \( t = \delta/\omega_0 \approx 0.0206 \text{ sec} \).
Observations:
• constant amplitude; reason, no way for the system to dissipate energy

• for a given mass \( m \) and spring constant \( k \) the system will always vibrate with the same frequency \( \omega_0 \).

• the initial conditions help determine the amplitude; recall that \( A = u_0 \) and \( B = v_0 / \omega_0 \) and \( R = (A^2 + B^2)^{1/2} \).

• since the period is given by \( T = 2\pi / \omega_0 = 2\pi (m/k)^{1/2} \) as \( m \) increases the period \( T \) increases, so larger masses vibrate more slowly.

• since the period is given by \( T = 2\pi / \omega_0 = 2\pi (m/k)^{1/2} \) as \( k \) increases (meaning the spring gets stiffer) the period \( T \) decreases which means the system vibrates more rapidly.
**Example:** (English units)

A weight of 4 lb stretches a spring 2 inches. The mass is displaced an additional 6 inches and then released. Construct the IVP for Undamped Free Vibration. *(Use feet for the linear measure.)*

There are four parameters that determine the IVP: mass, spring constant, and two initial conditions.

- **Weight:** \( w = mg \) (downward force)
- **Spring force:** \( F_s \) is proportional to the stretch of the spring from \( w = k * (\text{stretch amt}) \) (up or down force)

Mass = \( m = \frac{w}{g} = \frac{4 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{1}{8} \text{ lb·sec}^2/\text{ft} \)

From the information that a weight of 4 lb stretches a spring 2" = 1/6 ft we have \( k = \frac{4 \text{ lb}}{1/6 \text{ ft}} = 24 \text{ lb/ft} \)

From the information that the mass is displaced an additional 6" and then released we have \( u(0) = 6 \text{ "} = \frac{1}{2} \text{ ft} \) and \( u'(0) = 0 \) (since the mass is just released.)

IVP: \( mu'' + ku = 0, \ u(0) = u_0, \ u'(0) = v_0 \) is

\[(1/8)u'' + 24u = 0, \ u(0) = 1/2, \ u'(0) = 0\]
Find the solution of IVP \((1/8)u'' + 24u = 0\), \(u(0) = 1/2\), \(u'(0) = 0\) \[m = 1/8 \text{ lb} \cdot \text{sec}^2/\text{ft}\] \[k = 24 \text{ lb/ft}\]

The general form of the solution is \(u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)\)

We define \(\omega_0^2 = k/m\) so \(\omega_0^2 = 24/ (1/8) = 192\), and \(\omega_0 = (192)^{1/2} \approx 13.856\)

Using the initial conditions we find that \(A = u_0\) and \(B = v_0/\omega_0\) so \(A = \frac{1}{2}\) and \(B = 0\).

Thus \(\delta = \tan^{-1}(B/A) = 0\).

So we have \(u(t) = \frac{1}{2} \cos(13.856t)\).

**Amplitude** = \(R = \frac{1}{2}\)

**Natural frequency (or circular frequency)** = \(\omega_0 \approx 13.856\)

**Period of motion** = \(T = 2\pi/\omega_0 = 2\pi(m/k)^{1/2} \approx 0.453\) sec

Note the short period. There was a small mass & stiff spring. So we have rapid vibration. If we divide \(\omega_0\) by \(2\pi\) we get the number of cycles per second which is about 2.2 here. (This called the frequency.)
The **kilogram** (symbol: kg) is the base unit of **mass** in the International System of Units (SI) which is the modern standard governing the metric system.

The **newton** is the SI unit for **force**; it is equal to the amount of net force required to accelerate a mass of one kilogram at a rate of one meter per second per second squared.

A body with a mass $m = \frac{1}{2}$ kilogram (kg) is attached to the end of a spring that is stretched 2 meters (m) by a force of 100 newtons (N). It is set in motion with initial position $u_0 = \frac{1}{2} m$ and initial velocity $u'(0) = -10$ (m/s). (Note this means it is given an upward velocity.) Find an expression for $u(t)$ the position of the object at time $t$; determine the amplitude, circular frequency, period, and phase angle.

Mass $= m = \frac{1}{2}$ kilogram (kg)  
Spring constant $= k = \frac{(100 \text{ N})}{(2 \text{ m})} = 50 \text{ N/m}$

The DE is $(1/2) u'' + 50 u = 0$  
Initial conditions are $u(0) = \frac{1}{2} m$, $u'(0) = -10$ m/s

We define $\omega_0^2 = k/m$ so $\omega_0^2 = \frac{50}{(1/2)} = 100$, and $\omega_0 = 10$ (rad/s)

Using the initial conditions we find that $A = u_0$ and $B = \frac{v_0}{\omega_0}$ so $A = \frac{1}{2}$ and $B = \frac{-10}{10} = -1$.

So we have $u(t) = \frac{1}{2} \cos(10 t) - \sin(10 t)$.

Ref: Edwards & Penney
For \( u(t) = \frac{1}{2} \cos(10t) - \sin(10t) \) we have

Amplitude \( R = ((1/2)^2 + (-1)^2)^{1/2} = (1/2)^{5^{1/2}} \approx 1.12 \text{ m} \)

Natural frequency (or circular frequency) = \( \omega_0 = 10 \)
so the frequency is \( 10/2\pi \approx 1.59 \) cycles per second

Period = \( T = 2\pi/\omega_0 \approx 0.63 \text{ sec} \)

To determine the phase angle we use

\[
\begin{align*}
  u(t) &= R \left( \frac{A}{R} \cos(\omega_0 t) + \frac{B}{R} \sin(\omega_0 t) \right) = R (\cos(\delta) \cos(\omega_0 t) + \sin(\delta) \sin(\omega_0 t)) \\
  \cos(\delta) &= \frac{A}{R}, & \sin(\delta) &= \frac{B}{R} \\
  \cos(\delta) &= \frac{1}{\sqrt{5}}, & \sin(\delta) &= \frac{-2}{\sqrt{5}}
\end{align*}
\]

Thus using values of \( A = 1/2, B = -1, \) and \( R = \frac{1}{2} \sqrt{5} \) we get

\[
  u(t) = \frac{\sqrt{5}}{2} \left( \frac{1}{\sqrt{5}} \cos(10t) - \frac{2}{\sqrt{5}} \sin(10t) \right) = \frac{\sqrt{5}}{2} \cos(10t - \delta)
\]

Using the theory for determining the shift \( \delta \) using \( \tan(\delta) = B/A \) with the fact that \( A > 0 \) and \( B < 0 \) we have

\[
  \delta = 2\pi + \tan^{-1}(-1/1/(1/2)) = 2\pi - 1.1071487 \approx 5.1760 \text{ (rad)}
\]

Finally

\[
  u(t) = \frac{\sqrt{5}}{2} \cos(10t - 5.1760)
\]

The phase shift is \( \delta/\omega_0 \approx 0.51760 \)
\[ u(t) = \frac{\sqrt{5}}{2} \cos(10t - 5.1760) \]
A very nice applet to view an undamped vibrating mass is available
http://ngsir.netfirms.com/englishhtm/SpringSHM.htm. In this applet the upward direction is considered positive and downward negative. There is no way in this applet to impart an initial velocity and k is fixed. However, you can change the mass which will change the period.

This applet may not work on your machine because of JAVA settings.
$k = 3 \text{ N/m}, T = 2.57 \text{ s}$

Mass = 500 g

- Circular Motion
- Graphs

Step
Continue
Pause
Reset
Timer/Start

- Spring-mass only
  - Displacement
  - Velocity
  - Acceleration

Time = 0 s

Spring-mass oscillator
Spring-mass oscillator

- Mass = 500 g
- Spring constant k = 3 N/m
- Period T = 2.57 s

Graph options:
- Circular Motion
- Graphs
- Step
- Continue
- Pause
- Reset
- Timer/Start

Graph types:
- Displacement
- Velocity
- Acceleration
Example of a different kind.

An unexpected case for engineers to consider in their design and operation of an unloading process.

A truck is unloading a heavy machine weighing 800 lb. by a crane. The (elastic) cable is suddenly seized (jammed) at time $t$ from a descending velocity of $v = 20$ ft/min.

Because the machine is attached to an elastic cable, which has the characteristics of a "spring," we may simulate this situation to a simple mass-spring systems.

(a) Set up the IVP. (Beware of units.)
(b) Solve the IVP. (Solution is $u(t) = 0.0062113 \sin(53.666 \ t)$; use $g = 32$.)
(c) Find the maximum tension in the cable induced by the vibrating machine. The maximum tension in the cable is determined with the maximum total elongation of the steel cable times $k + \text{weight}$.
(d) The maximum stress in the cable which is a stranded steel cable is 0.5 inch in diameter is 40,000 psi. Will the cable break? (Stress = maximum tension/cross sectional area of the cable.)

Adapted from work by Dr. Tai-Ran Hsu at SJSU.