Introduction to Second Order Linear DEs and IVPs

Key terms and Ideas:

• General form of a 2\textsuperscript{nd} order DE

• Linear 2\textsuperscript{nd} order DEs
  • Homogeneous form
  • Nonhomogeneous form

• Principle of Superposition

• Existence and Uniqueness of Solutions

• Constructing/Building solutions
General form of a 2nd order DE:

\[
\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)
\]

Some function involving independent variable \(t\), dependent variable \(y\) and the derivative of \(y\).

Usually, we will denote the independent variable by \(t\) since time is often the independent variable in physical problems, but sometimes we will use \(x\) or \(\theta\) instead.

Nonlinear model for the motion of a simple pendulum. (Note \(\sin \theta\).) Linear model for the motion of a simple pendulum. It assumes angle \(\theta\) is small. So \(\sin(\theta)\) is approximately \(\theta\).

Nonlinear DE because of the product of \(y\) and \(y'\).

\[
\frac{d^2y}{dt^2} + ty = t^2
\]

Linear DE
STANDARD form for a 2nd order Linear DE:

\[ y'' + p(t)y' + q(t)y = g(t) \]

If \( y'' \) has a coefficient we can divide each term by that expression to get the coefficient of \( y'' \) to be 1.

Note: the coefficients and right side are strictly in terms of independent variable \( t \).

The DE is called **homogeneous** if \( g(t) = 0 \) otherwise it is called **nonhomogeneous**.

\[ y'' + p(t)y' + q(t)y = 0 \]

Homogeneous DE.

We will see that "**IF** we can solve the homogeneous DE \[ y'' + p(t)y' + q(t)y = 0 \] then we will be able to solve the **nonhomogeneous DE** \[ y'' + p(t)y' + q(t)y = g(t) \]

So solving nonhomogeneous DEs requires **two steps**: solve the corresponding homogeneous DE and use that solution to assist in solving the nonhomogeneous DE.

If we denote the solution of the homogeneous DE as \( y_h(t) \) and use \( y_p(t) \) to denote some solution of the nonhomogeneous DE, the set of all solutions of the original nonhomogeneous DE \[ y'' + p(t)y' + q(t)y = g(t) \] is given by \( y(t) = y_h(t) + y_p(t) \).
Special property of homogeneous linear DEs:

If $y_1(t)$ and $y_2(t)$ are two solutions of the homogeneous linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

then the linear combination $C_1 y_1(t) + C_2 y_2(t)$ is also a solution for any values of the constants $C_1$ and $C_2$.

If $\sin(2t)$ and $\cos(2t)$ are solutions of

$$y'' + p(t)y' + q(t)y = 0$$

so are each of the following expressions:

$34\sin(2t)$, $-6.78\cos(2t)$, and $9\sin(2t) - 7\cos(2t)$.

This is called the property Principle of Superposition.

**WARNING:** This property is NOT true for nonhomogeneous DEs.
When we solve DEs we get a **family of solutions**. When we solve IVPs we get **one solution** obtained by imposing the initial conditions on the family of solutions.

**Existence and Uniqueness Theorem for Linear 2nd order IVPs**

Consider the initial value problem  
$$y'' + p(t)y' + q(t)y = g(t), \ y(t_0) = y_0, \ y'(t_0) = y'_0$$
where $p$, $q$, and $g$ are **continuous** on an open interval $I$ that contains $t_0$. Then there is **exactly one solution** $y = \phi(t)$ of IVP, and the solution exists throughout the interval $I$.

We emphasize that the theorem says **three things**:

1. The initial value problem **has a solution**; in other words, a **solution exists**.

2. The initial value problem has **only one solution**; that is, the **solution is unique**.

3. The solution $y = \phi(t)$ is **defined throughout the interval $I$ where the coefficients are continuous and is at least twice differentiable there**.

**WARNING:** For most nonhomogeneous problems of the form $y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0, \ y'(t_0) = y'_0$ it is **not** possible to write down a useful expression for the solution.

**This is a major difference** between first order and second order linear equations.
The **crucial step** in solving DE \( y'' + p(t)y' + q(t)y = g(t) \) or IVP \( y'' + p(t)y' + q(t)y = g(t), \ y(t_0) = y_0, \ y'(t_0) = y'_0 \) is to **find the family of solutions to the corresponding homogeneous DE** \( y'' + p(t)y' + q(t)y = 0 \).

Here is where some theory helps. What follows is a summary. Details are in Section 3.2 of the text.

**First** find two (different) solutions \( y_1(t) \) and \( y_2(t) \) of DE \( y'' + p(t)y' + q(t)y = 0 \).

**Next** we must check that the set of solutions \( C_1y_1(t) + C_2y_2(t) \), where \( C_1 \) and \( C_2 \) are any constants, includes all possible solutions to the DE. To do this we appeal to some theory that says

\[ C_1y_1(t) + C_2y_2(t) \text{ includes all solutions to the DE if and only if } W = y_1(t)y'_2(t) - y'_1(t)y_2(t) \neq 0. \]

\( W \) is called the **Wronskian** of functions \( y_1(t) \) and \( y_2(t) \). If the Wronskian is not zero we say that the pair of solutions \( y_1(t) \) and \( y_2(t) \) forms a **fundamental set of solutions**.

This means all solutions of the DE can be produced using the **Principle of Superposition** using \( y_1(t) \) and \( y_2(t) \).

**It sounds EASY!** \[ \text{But what's the “CATCH”?} \]

There is **no general way** to find \( y_1(t) \) and \( y_2(t) \) for every DE \( y'' + p(t)y' + q(t)y = 0 \). **But there are some special types** of DEs where it is easy to find fundamental sets. **So we need to consider special types of DEs.**
**Definition:** For a pair of differentiable functions $y_1(t)$ and $y_2(t)$

$$W(y_1(t), y_2(t)) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{bmatrix} = y_1(t)y'_2(t) - y'_1(t)y_2(t)$$

$W$ is called the **Wronskian** of the pair of functions and $\det$ is the abbreviation for determinant of a matrix.

An easy way to remember the computation of the Wronskian is shown in the following diagram.

Note the minus sign!