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Equity Considerations in Traditional Full Cost Allocation Practices:
An Axiomatic Perspective

INTRODUCTION

Cost accounting serves several purposes, each requiring different—perhaps inconsistent—cost information. For instance, economic decisions such as make-or-buy, capacity planning or product pricing may require different costs to be considered. Furthermore, cost accounting information may also be related to the reward structure in decentralized organizations in order to motivate decisions by managers of profit centers that are congruent with the overall objectives of the firm. A cost accounting system is required to serve all these purposes as the occasion may demand.

The preceding examples are oriented toward economic efficiency as a guiding principle. In other situations such as regulatory price setting, inventory valuation for tax purposes and determination of costs for cost-type contracts, there may also be an emphasis on cost allocations that are "fair" and "equitable". The same is true when costs are to be assigned to different organizational entities in a manner that will enhance continued collaboration (or even co-existence) over protracted periods of time. This is especially true when full cost, as distinct from incremental cost, allocations are to be effected.

Even when a particular cost allocation scheme induces a Pareto efficient outcome, it may not distribute full costs to the users in a manner that is perceived to be "fair" and "equitable", and further "side-payments" may be required to achieve a distribution of costs that satisfies this criterion of fairness. In the absence of a consensus on the specification of an "aggregate preference function", however, the concepts of "fairness" and "equity" remain ambiguous. Several new approaches to this problem have been discussed in the recent accounting literature. We shall try to approach this topic in a way that will enable us to relate those ideas with traditional cost accounting practices.

*The helpful comments and suggestions of W.W. Cooper and C.J. Christenson are gratefully acknowledged.
Thus, for instance, Shubik (1962), Spinetta (1975), Jensen (1977), Callen (1978) and Roth and Verrecchia (1979) have suggested the use of the Shapley Value, a prominent solution concept in the theory of games, for allocating common costs and have justified it on the basis of the fact that it is uniquely determined by a set of axioms. This is in contrast to others like A. Thomas (1969 and 1974) who assert that all cost allocation methods are arbitrary and no one allocation scheme can be defended against all others.

In this paper, we shall take a different approach and attempt to develop a set of axioms from which the usual cost accounting practices can be derived. Axiomatic formulations can be (and have been) developed for various purposes. They might be used to develop a new concept as Shapley (1953) did in suggesting the Shapley Value as a new solution concept for n-person cooperative games. These formulations might then be drawn upon and used in other applications as in the proposed uses of Shapley Value approaches to cost allocation problems that have been referred to above.

This is not the only way in which axiomatic formulations have been used. They can also be used for simplification and clarification as in Peano's axiomatization of arithmetic and this extends to pedagogical uses as in Euclid's use of the axiomatic approach as a basis for his text on geometry. Axiomatic formulations can also be used to describe existing levels of practice (and theory) as in Woodger's attempt to axiomatize biology, Hull's attempt to axiomatize psychology, and Kohler's attempt to axiomatize accounting. This approach can also be used to unify a variety of disciplines as in the attempts of Hilbert, Ackermann, et al., to axiomatize all of mathematics—which Gödel showed to be impossible of achievement.

As an offshoot of these latter developments, one may also try to axiomatize common practices and then to use this as a basis for identifying proposals which are or are not consistent with these practices. In common with all of the other uses of axiom systems, one thus tries to reduce these practices to the simplest possible bases in order to assess their common properties and what may be removed or altered in these practices when one or more of these axioms is altered. Furthermore, one can then also identify seemingly removed practices or theories which also relate to these axioms with the result that either these practices will also change or else the indicated connection will be lost when one or more of the axioms is changed—e.g., for supposedly "improved" approaches to some set of problems.

This last paragraph indicates the approach that we shall employ in this paper. We shall illustrate it by reference to commonly employed practices in cost accounting and how these might be affected if one or more of the alternative cost allocation approaches now being suggested are employed. We shall also show how our axiom system relates to other developments such as Lev and Theil's (1978) approach to depreciation via entropy theory or Bell, Keeney and Little's (1975) axiomatic formulation of market share analyses.

None of the above is intended to mean that proposed approaches such as
Roth and Verrecchia's (1979) use of the Shapley Value concept is wrong for cost allocation. It only means that one should be aware of the consequences of using them in general or in conjunction with the traditional cost allocation methods—and our axiomatic development makes this easy to do. We can also use our axiomatic framework to develop alternative uses of some of these ideas (such as the Shapley Value) in different parts of cost-allocation practice where they can effect solutions without altering other parts of practice.

This will be done in the sections that follow. First, we shall consider a simple situation and indicate certain inadequacies and shortcomings of the Shapley Value in its applications to the cost allocation problem. Then we shall suggest modifications in the Shapley Value axioms and demonstrate that these revised axioms also uniquely determine an allocation scheme. This will enable us to identify the axioms underlying most of the traditional allocation practices in cost accounting. It will also thereby show that the Shapley Value axioms as applied to cost allocation problems by Roth and Verrecchia, for instance, are inconsistent with the axioms underlying traditional cost accounting.

From this vantage point we will then make contact with other approaches such as the maximum entropy criterion suggested by Lev and Theil (1978), the core criteria suggested by Hamlen, Hamlen and Tschirhart (1977), and the allocation scheme suggested by Moriarity (1975). Checking for consistency between these other approaches and our axiom system will also help us to identify differences and similarities. In this manner, our axiomatic framework will enable us to integrate principles underlying cost allocation schemes with other approaches suggested in the recent accounting literature. In this context, it is well to quote Ijiri (1967) who states:

Accounting has its own way of thinking about, observing, and organizing business phenomena. What is more important, accounting has its own discipline and own philosophy, which have been developed over may centuries. This does not mean that they should not be changed. It emphasizes that the response to the challenges should be made keeping always in mind the effects of this response upon accounting foundations. Otherwise, accounting will soon become simply a patchy collection of practices.

Shortcomings of Shapley Value Allocation Schemes

In this section, we shall describe shortcomings of the Shapley Value in the context of a simple situation involving cost allocations. Consider a service department providing a single homogeneous service to \( j = 1, \ldots, n \) user departments. Let the amount of service provided be measured by a variable \( q \), which is restricted to take non-negative rational values. Let the cost function be represented by \( c(q) \), which is assumed to be strictly increasing.
Further, let $q_j$ be a measure of the service provided to each of the $j = 1, \ldots, n$ users. Setting $q = \sum_{j=1}^{n} q_j$, we can then write $c = c(\sum_{j=1}^{n} q_j)$ to represent the total cost of providing the common service. An allocation scheme is then defined to be a function $x$ mapping each user department $j$ (using $q_j$ units of the common service), to an allocated cost $x_j$, with $\sum_{j=1}^{n} x_j = c$ when a full cost allocation is to be made.

In the context of this cost allocation problem, a Shapley Value allocation scheme would allocate to user department $k$ an amount $x_k$ given by:

$$x_k = \sum_{S \subseteq N \ni k} \frac{(n-s)!}{n!} (s-1)! c(\sum_{j \in S} q_j) - c(\sum_{j \in S - \{k\}} q_j)$$

where $N = \{1, \ldots, n\}$

The Shapley Value is uniquely determined by a set of three axioms which can be stated as follows:

**Axiom S1** Full Cost Allocation

The costs allocated to the user departments add up to the total cost of providing the service,

i.e. $\sum_{j=1}^{n} x_j = c$

**Axiom S2** Symmetry

If the amount of service provided to two user departments is the same, then the costs allocated to them must be the same,

i.e. $q_f = q_g \Rightarrow x_f = x_g$

The assumption that $c(q)$ is strictly increasing insures that for any $S \subseteq N - \{f\} - \{g\}$:

$$q_f = q_g \Rightarrow c(\sum_{j \in S \cup \{f\}} q_j) = c(\sum_{j \in S \cup \{g\}} q_j)$$

The above two axioms are relatively easily accepted if one wants to accept the principle of full cost allocation (sharing of total burden) among all participants in a relatively even manner. However to complete the Shapley Value axioms we also need:

**Axiom S3** Additivity of Cost Allocation Problems

If $v$ and $w$ are the cost functions for two services with quantity measures $q^v$ and $q^w$ respectively, such that $v(\sum_{j \in S} q^v_j) + w(\sum_{j \in S} q^w_j) = c(\sum_{j \in S} q_j)$ for $S \subseteq N$, then the allocation
scheme applied to \( v \) and \( w \) separately yielding \( x^v_j \) and \( x^w_j \) must add up to \( x_j \) obtained by applying the allocation scheme to \( c \).

This axiom requires, for example, that if the costs of a common service are disaggregated into fixed capacity costs and variable operating costs, and if these fixed and variable costs are allocated independently, then the total cost allocation (i.e., the sum of the independent allocations of fixed and variable costs) to each user should be no different from the allocations that would be obtained from a direct allocation of all the costs, ignoring the distinction between the fixed and variable cost components. The axiom further requires this "additivity" relation to be satisfied in all possible situations—irrespective of the variability in the capacity commitments or demand fluctuations of the user departments.

The Shapley Value axioms are not free from criticism. In particular, this last "additivity" axiom has been considered questionable. In this context, Luce and Raiffa (1957), for instance, state:

> The last condition is not nearly so innocent as the other two. For although \( v + w \) is a game composed from \( v \) and \( w \), we cannot in general expect it to be played as if it were the two separate games. It will have its own structure which will determine a set of equilibrium outcomes which may be very different from those for \( v \) and for \( w \). Therefore, one might very well argue that its \textit{a priori} value should not necessarily be the sum of the values of the two component games. This strikes us as a flaw in the concept of value, but we have no alternative to suggest.

A further possible shortcoming of the Shapley Value allocation scheme that incorporates this additivity axiom is that it makes the allocation sensitive to the way cost centers are used or organized. As shown in the following example, the allocations can differ significantly if two cost centers are merged and considered as a single entry.

Consider three departments \( A, B \) and \( C \) that require the use of a service in amount \( q \). The cost function for this service is given by

\[
c(q) = \begin{cases} 
2q & \text{for } 0 \leq q \leq 60 \\
3q/2 + 30 & \text{for } 60 \leq q \leq 80 \\
q + 70 & \text{for } 80 \leq q
\end{cases}
\]

The amount of service required by departments \( A, B \) and \( C \) are

\[q_A = 50, \quad q_B = 30, \quad \text{and} \quad q_C = 30.\]

Therefore we have,

\[c_A = 100, \quad c_B = c_C = 60,\]
Evidently the least cost route is for the three departments to jointly procure the service for 180 dollars. The Shapley Value would then allocate these common costs to the three departments as follows:

Department A: 100 - \([0 + 10/6 + 10/6 + 40/3]\) = $83.34
Department B: 60 - \([0 + 0/6 + 10/6 + 30/3]\) = $48.33
Department C: 60 - \([0 + 0/6 + 10/6 + 30/3]\) = $48.33

Now suppose Departments B and C are merged, and considered as one department, say BC, for the purpose of the cost allocation. No overall saving is involved, however, as the total costs continue to be $180. The Shapley Value would now allocate the costs as follows:

Department A: 100 - \([0 + 40/2]\) = $80
Department BC: 120 - \([0 + 40/2]\) = $100

Thus we note that the costs allocated to department A decrease from $83.34 to $80.00, not because of any effort or activity on its part but simply because the other two departments, B and C, were considered as a single department BC. This is an awkward result from the standpoint of a practical cost allocation scheme since the motivation for such an adjustment is not apparent. In any case, it also illustrates the insensitivity of the Shapley Value allocations to the way in which cost centers are organized. This suggests altering or revising this aspect of the Shapley Value axioms to render them suitable for practical use in cost allocations.

An Alternative Set of Axioms

In this section, we propose to replace the "additivity of games" axiom by another axiom requiring "additivity of cost centers." We shall also show that these revised axioms preserve a property of the Shapley Value axioms, viz., that our revised axioms are necessary and sufficient to uniquely determine an allocation scheme.

For this purpose, we write our axioms as follows:

Axiom A1 Full Cost Allocation

\[ \sum_{j=1}^{n} x_j = c \]

Axiom A2 Symmetry

\[ q_f = q_g \rightarrow x_f = x_g \]

Axiom A3 Additivity of Cost Centers

If a cost center \( k \) is subdivided into two cost centers \( f \) and \( g \), such
that \( q_k = q_f + q_g \); then the costs allocated to each of the remaining cost centers remains the same as before when \( k \) was considered as a single entity.

In other words, we do not want the allocation to change unless or until the amount of service usage changes with the reorganization.

We shall next provide a simple but complete proof to establish that these three axioms uniquely determine an allocation scheme. We shall proceed by proving two basic lemmas before stating the uniqueness theorem.

**Lemma 1:** If \( f, g \) and \( h \) are three users such that the amount of service provided to \( h \) equals the sum of the amounts of service provided to \( f \) and \( g \), then the costs assigned to \( h \) must equal the sum of the costs assigned to \( f \) and \( g \), i.e.,

\[
q_h = q_f + q_g \implies x_h = x_f + x_g
\]

To prove this lemma we consider a related situation where \( f \) and \( g \) are considered to be a single entity \( k \), such that \( q_k = q_f + q_g \). For this situation, we denote by \( x \) an allocation scheme that satisfies axioms A1 to A3.

Then by virtue of axioms A1 and A3, we have,

\[
x_h = x_h
\]

and

\[
x_k = c - \sum_{j \neq f, g} x_j
\]

\[
= c - \sum_{j \neq f} x_j
\]

\[
= x_f + x_g
\]

Also, using axiom A2 we write

\[
x_h = x_k
\]

Therefore, \( x_h = x_h = x_k = x_f + x_g \)

**Lemma 2:** If there are \( p \) users, each being provided the same amount of service, then the costs allocated to each such user is \( c/p \).

This lemma is an immediate consequence of the symmetry axiom. Since the amount of service provided to each user is the same, the costs assigned to each of them should also be the same, say \( x_o \). Hence, the total costs assigned are \( p \cdot x_o \), which because of the full cost allocation axiom should equal \( c \), the total cost of providing the common service.

i.e., \( p \cdot x_o = c \)
Therefore, $x_0 = c/p$.

Now we state and prove the following uniqueness theorem.

**Theorem:** For a given set of rational numbers \( \{q_j\}_{j=1}^{n} \) representing the amounts of service provided to each of \( j = 1, \ldots, n \) users, there exists a unique allocation scheme \( x \) that satisfies axioms A1 to A3. This allocation scheme is given by

\[
x_j = \frac{q_j}{\sum_{k=1}^{n} q_k} \cdot c
\]

**Proof:** We begin by noting that since each \( q_j \) is a rational number, it can be represented as \( (a_j/p) \cdot q \) where \( p > 0 \) and \( a_j \) are non-negative integers such that \( \sum_{j=1}^{n} a_j = p \), and \( q = \sum_{j=1}^{n} q_j \).

Now, in order to prove this theorem, we consider first a related situation where we have \( p \) users, each being provided an amount of service equal to \( q/p \). Then by virtue of lemma 2, the costs assigned to each user must be \( c/p \).

Next we consider a group of \( a_1 \) of these users as a single entity. By virtue of lemmas 1 and 2, the costs assigned to this single entity \( (j = 1) \), are given by

\[
x_1 = a_1 \cdot \frac{c}{p}
\]

Further, by sequentially repeating this argument, we have,

\[
x_j = a_j \cdot \frac{c}{p} = \left(\frac{q_j}{q}\right) \cdot c \quad \text{for each } j = 1, \ldots, n
\]

We have thus shown that our three axioms are sufficient to determine a unique allocation scheme, since any allocation scheme that satisfies the three axioms must be given by \( x_j = (q_j/q) \cdot c \) for each \( j = 1, \ldots, n \). This in turn raises the question of whether all of our axioms are needed for this or whether a subset will do. In the Appendix, we demonstrate by means of counter-examples that no two of the axioms are sufficient to uniquely determine an allocation scheme, thus implying that this set of three axioms is also a set of necessary axioms.

Our axioms are thus necessary and sufficient to uniquely determine an allocation scheme. We shall next relate our, thus determined, allocation scheme to the allocation practices that are commonly employed. As can be seen, it includes the customary cost allocation schemes which allocate the costs of a common facility in the proportion of its usage. Thus, our axioms are broad enough to include all the traditional cost allocation methods for which (a) common bases are determined to measure the amount of service provided to each user, and (b) the allocation proceeds by assigning the costs...
to each user in proportion to this measure for that user. Thus, for instance, we would cover allocation schemes in which machine overheads are allocated on the basis of machine-hours and schemes in which steam generation costs are allocated in the proportion of the steam utilization by each department.

Having indicated some (but not all) of what our axioms cover, we should also indicate some of what they do not cover. Note, for instance, that the properties of our third axiom differ fundamentally from the third of the Shapley Value axioms as interpreted, for instance, by Roth and Verrecchia. Since we have already proved the necessity and sufficiency of our axioms, and further identified them with traditional cost allocation schemes, we can assert that these Shapley Value allocation schemes are, in general, not consistent with the extensively used traditional allocation schemes.

Of course, this does not complete what can be done in the way of comparisons. In the next two sections, we shall examine some other recent approaches which have also sought to identify or explain—without necessarily trying to improve (or resolve)—the commonly used approaches in accounting. This will include the information theoretic approach, via entropy concepts, employed by Lev and Theil, as well as the attempts to characterize or interpret accounting allocation processes in terms of the concept of a core from game theory.

**Maximum Entropy Approach**

In this section, we demonstrate that the allocation scheme derived from our axiom set also satisfies a criterion suggested by Lev and Theil (1978). They suggest the use of the maximum entropy criterion for the selection of a depreciation (allocation over time) scheme under conditions of imperfect information. For this purpose, *all* the relevant information that is available is represented by constraints placed on the cost distribution (or allocation) function. Since *all* the available information is so represented, the distribution (or allocation) function is then selected as the one which maximizes entropy. From an information theoretic perspective, this is equivalent to (a) making the fullest possible use of the available information as impounded in the constraints, and (b) minimizing the additional information content which the distribution scheme requires for its justification. In this context, Lev and Theil note:

A distribution selected by the maximum entropy criterion will thus represent the least prejudiced description of the state of knowledge; it is consistent with what is known but expresses "maximum uncertainty" (ignorance) with respect to all other matters which are, by assumption, unknown.

Lev and Theil use the maximum entropy criterion to select a depreciation
scheme. They find that the straight-line depreciation scheme maximizes entropy when we know (a) the service life of the asset and (b) the initial and terminal values of the asset—and nothing else. When the service life of the asset is not known, but the mean service-use period is known, they find that an exponential depreciation scheme maximizes entropy. Undoubtedly if one is willing to make more assumptions when one has more knowledge, then other distributions may emerge.

We proceed in a similar vein but apply the same criterion of maximizing entropy not to depreciation per se but to a situation in which the costs of a common facility are to be allocated to the several users. These are of interest, however, in that these, too, represent common cost allocation problems, involving some knowledge about the usage pattern. We assume that the only information available is the total cost of providing the common service, and the measures of the service utilization for each user. In this situation, the entropy measure is expressed as

$$H = -\int_0^Q x(u) \log x(u) \, du,$$

where the variable $u$ represents the service provided by the common facility and $x(u)$ is the cost distribution function to be determined. We note that the value of this cost distribution function is known to be zero outside the interval $[0,Q]$, and $x(u)$ is the function to be determined.

To maximize the entropy, subject to the full cost allocation constraint, we write,

$$\text{max } H = -\int_0^Q x(u) \log x(u) \, du$$

Subject to

$$\int_0^Q x(u) \, du = c.$$

We proceed to pointwise maximize the integrand. The relevant theorem that we employ [see e.g. Theorem 5.1 on p. 215 and Example 6 on p. 217 in Hestenes (1966)] states that $x^*$ will maximize $J_0(x) = \int_a^b F_0(u,x(u)) \, du$ on the class of admissible functions $x$ that satisfy $J_1(x) = \int_a^b F_1(u,x(u)) \, du = 0,$ only if there exist multipliers $w_0 \geq 0$ and $w_1$ (not both zero) such that the inequality $F(u,x(u)) \leq F(u,x^*(u)), a \leq u \leq b,$ holds for all admissible elements $(u,x)$; where $F = w_0 F_0 + w_1 F_1$.

Returning to our proposition, we seek the function $x = x^*$ which maximizes $J_0(x) = -\int_0^Q x(u) \log x(u) \, du$ subject to $J(x) = c - \int_0^Q x(u) \, du = -\int_0^Q (x(u) - c/Q) \, du = 0.$ Let $F(u,x) = -w_0 x(u) \log x(u) - w_1 (x(u) - c/Q).$ If $w_0 = 0,$ we can have $-w_1 (x^* - x) \geq 0,$ for all admissible $x, x(u) \geq 0, 0 \leq u \leq Q,$ only if $w_1 = 0.$ But the condition of maximization is that both $w_0$ and $w_1$ cannot be zero. Hence, $w_0 \neq 0,$ which implies $w_0 > 0$ and we can choose $w_0 = 1.$

With $w_0 = 1,$ however, we then have
\[ F(u, x) = -x(u) \log x(u) - w_1(x(u) - c/q). \]

Since \( x^* \) maximizes \( F(u, x) \) for each given \( u \), we must have \( F'_x = \frac{\partial F}{\partial x} = 0 \), when \( x = x^* \). This in turn gives \( 0 = \frac{\partial F}{\partial x} = -\log x - 1 - w_1 \). Choosing \( w_1 = w \) it follows that \( x^*(u) = e^{-w-1} \), which is independent of \( u \) and hence a constant function, \( x(u) = c/q \).

Therefore, the cost allocated to a user \( j \), being provided an amount of service equal to \( q \) is given by:

\[
x_j = \int_{q}^{x_j} x(u) \, du
= \int_{q}^{x_j} (c/q) \, du
= q_j \cdot (c/q)
\]

Thus we see that the allocation scheme that maximizes entropy is given by \( x_j = (q_j/q) \cdot c, j = 1, \ldots, n \), which is the same as the allocation scheme derived from our axiom set in the preceding section.

In this fashion, we have shown that our axiom system encompasses the maximum entropy criterion suggested by Lev and Theil, when extended to the situation of allocation of costs of a common facility to user departments. We also need to note that our axiom system does not extend to the Lev and Theil exponential scheme. This would require a different set of axioms. But, in any case, the question of internal allocation of the costs of a common service does not commonly encounter the equivalent of an unknown useful life of a depreciable asset, and so such an extension does not appear warranted for the practices being studied here.

**Extension to Other Approaches to Cost Allocations**

Having established contact with the information theoretic approach of Lev and Theil, we next turn to the game theoretic approach adopted by Hamlen, Hamlen and Tschirhart (1977). They suggest that a cost allocation scheme must represent a solution lying in the core of the \( n \)-person game corresponding to the cost allocation problem. The satisfaction of the core conditions is considered necessary not only for equity considerations, but also for the continuation of the coalition (by analogy with \( n \)-person games).

We summarize the core conditions verbally as follows. These conditions require that no individual user and no group of users should be allocated more costs than what that user or group of users would have to bear if they had decided to obtain the same amount of service independently from an alternative source. When the cost function for the common service is the same for all the users and when the marginal costs are positive and diminishing, our allocation scheme satisfies the core conditions.

A simple proof is provided for this proposition by Hamlen, Hamlen and Tschirhart (1977, p. 621), which we summarize as follows. Let \( S \) be a subset
of \( N = \{1, \ldots, n\} \), and let \( c(q) \) represent the cost function. Now since the marginal costs are positive and diminishing, the average costs also decrease as the coalition grows larger. We therefore have,

\[
\frac{c(\sum_{j \in N} q_j)}{\sum_{j \in N} q_j} \leq \frac{c(\sum_{j \in S} q_j)}{\sum_{j \in S} q_j}
\]

so that multiplying both sides by the denominator on the right,

\[
\sum_{j \in S} q_j \cdot c(\sum_{j \in N} q_j) \leq c(\sum_{j \in S} q_j)
\]

Thus, we see that the axioms underlying traditional cost allocation schemes are consistent with the core conditions when the marginal costs are positive and diminishing. Our axioms therefore preserve one more property of the Shapley Value axioms. Like the Shapley Value, the allocation scheme determined by our axioms satisfies the core conditions when the game is "convex".

We shall next examine yet another approach to cost allocation that has been suggested in the recent accounting literature. Moriarity (1975) suggests that the costs of a common facility be allocated in proportion to the costs of the best alternative for each user. If we interpret the basis measures \( q \) in our axiom system as "the costs of the best individual alternative," then the allocation scheme that would be uniquely determined by the three axioms is equivalent to Moriarity's allocation scheme.

A different interpretation of the term "basis measure" (service measure), a "primitive" in our axiom system, thus yields a different allocation scheme. In fact, in different situations different interpretations of the primitives (such as basis measure, cost function, etc.) may be appropriate, and each such interpretation may yield a different allocation scheme suitable for that situation. We may note that all these interpretations are isomorphic, and the allocation scheme in each case is determined by our uniqueness theorem. Our axiom system is thus "categorical", since a class of isomorphic allocation schemes is determined by these axioms. Euclidean geometry is "categorical" in the same sense: there can be more than one Euclidean space but all are isomorphic.

It is, of course, possible to construct alternative sets of axioms to derive our allocation scheme. Thus, for instance, we can adapt Bell, Keeney and Little's (1975, p. 137) axiomatic formulation of market shares to our problem of cost allocation as follows:

**Axiom M1**

\[
\sum_{j=1}^{B} x_j = c
\]

**Axiom M2**

\[
q_f = q_g - x_f = x_g
\]
Joint Cost Allocations

Axiom M3 \( x_j \geq 0, \quad j = 1, \ldots, n \)

Axiom M4 \( q_j = 0 \Rightarrow x_j = 0 \)

Axiom M5 Cost allocation to a given user will be affected in the same manner if the service usage of any other user is increased by a fixed amount \( \Delta \).

Bell, Keeney and Little prove that these axioms are sufficient to determine a unique allocation scheme. This allocation scheme is the same as the one determined by our axiom set. Thus, the Bell, Keeney and Little axioms provide an alternative basis for developing our allocation scheme. In the same vein, our axioms could be interpreted in terms of the market share problem of Bell, Keeney and Little, and employed to deduce their market share theorem. This also illustrates that it is possible to provide non-accounting interpretations for our axiomatic system, and derive "models" that are isomorphic to the cost allocation model.

Extension to Multiple Common Services

We have thus far considered only the case of a single common facility, and identified three axioms which yield the most common of traditional methods for allocating the costs of such a common facility. In this section, we shall examine the basic axioms underlying traditional methods for allocating costs in a more general situation involving more than one common facility. For this purpose, we shall first specify the following "primitive" terms for such an axiomatic system:

P1: Total Cost
The total cost in a given time period \( t \) is represented by \( c = c(t) \).

P2: Basic Activities
The basic activities are represented by the set \( M = \{j | j = 1, \ldots, n \} \).

P3: Cost Centers
The cost centers are represented by the set \( N = \{j | j = 1, \ldots, n \} \).

P4: Service Usage Measures
For each basic activity \( i \in M \), the amount of service provided in time period \( t \) to each cost center \( j \in N \), is represented by \( q_{ij}^t = q_{ij}^t(t) \). Further, we let \( q_i^t = \sum_{j=1}^{n} q_{ij}^t \).

P5: Activity Costs
The costs identified with basic activity \( i \in M \) in time period \( t \) is represented by \( c_i^t = c_i^t(t) \).
P6: Alternative Costs

The cost of the best alternative for providing the activities in S, where S is any non-empty subset of M, is represented by \( a_S = a_S(t) \).

Traditional methods of cost allocation require that (a) portions of the total costs are identified with each of the basic activities such that (b) these portions of costs identified with the basic activities add up to the total costs. In other words, it must be possible to uniquely and exhaustively identify the costs of providing each of the basic services. We formalize this assumption in terms of:

Axiom B1 Separability

It is possible to uniquely identify costs \( c_i \) with each of the basic activities such that they add up to the total costs;

i.e. \( \sum_{i=1}^{m} c_i = c \), where \( c \) is the total cost.

Implicit in this axiom is an assumption that the cost functions for the basic activities are separable, which is to say that each \( c_i \) is a function of the corresponding \( q_i \).

This axiom can then be adjoined to our earlier axioms to obtain what we require for identification with traditional cost allocation methods. To restate these axioms in a more suitable form, we shall first represent the cost allocation schemes as functions mapping each user \( j = 1, \ldots, n \) into an \( m \)-dimensional vector \( x_j = (x_j^1, \ldots, x_j^m) \) representing the costs of each of the \( i = 1, \ldots, m \) activities allocated to the user \( j \), so that the total costs allocated to the user \( j \) are \( x_j = \sum_{i=1}^{m} x_j^i \). Then our earlier full cost allocation axiom becomes:

Axiom B2 \( \sum_{j=1}^{n} x_j^i = c_i \) for each \( i = 1, \ldots, m \).

Our symmetry axiom becomes

Axiom B3 \( q_j^i = q_k^i \rightarrow x_j^i = x_k^i \) for each \( i = 1, \ldots, m \).

And our axiom on the additivity of cost centers is now extended to

Axiom B4 If a cost center \( k \) is subdivided into two cost centers \( f \) and \( g \) such that \( q_k^i = q_f^i + q_g^i \) for each \( i = 1, \ldots, m \), then the cost allocation vector \( x_j = (x_j^1, \ldots, x_j^m) \) for each of the remaining cost centers \( j \), remains the same as before—when \( k \) was considered as a single entity.
It is an immediate consequence of our earlier theorem that these four axioms uniquely determine an allocation scheme. Furthermore the extension is direct since the allocation scheme is now completely specified as:

\[ x^i_j = \frac{q^i_j}{\sum_{k=1}^{n} q^i_k} \cdot c^i \]

for each \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \).

It is interesting to note the similarity and the differences between Roth and Verrecchia's interpretation of the Shapley Value axioms and our axioms. Implicit in their interpretation of the Shapley Value axioms is the assumption that the cost centers (players of the game) are uniquely identifiable, and a subdivision of any activity into component activities before allocating costs, does not change the ultimate allocations. There is, so to speak, an interchange between the role of the cost centers and the activities in moving to our axioms. Thus we require the basic activities to be uniquely identifiable, and we also require that the subdivision of the cost centers does not change the ultimate allocations.

However suited for game theory purposes, the Shapley Value axioms for the cost allocation problem as formulated above are not consistent with traditional cost allocation methods. This follows, as previously noted, from our reformulation of these axioms in structuring our axiom system. Therefore, such Shapley Value allocation schemes cannot be simply adjoined to the extensively used traditional cost allocation schemes. Unless one is willing to discard or refute these traditional allocation schemes, the kind of approach proposed by Roth and Verrecchia, interpreting managers of cost centers as players in a n-person game, would not be admissible.

This however does not imply that the Shapley Value approach has no potential application for cost allocation problems. As we noted earlier in this section, it is a requirement for traditional cost allocation methods that the total costs be apportioned to each of the basic activities. (See the "separability" axiom.) Implicit in this was the assumption that the cost functions for the basic activities are separable. This creates problems in many practical situations involving jointness. If basic activities are identified at a level of aggregation that insures the separability of their cost functions, it becomes impossible to identify service measures that satisfy the remaining axioms. For example, consider a service department \( Z \) providing two services \( A \) and \( B \). One would generally disaggregate the costs to the level of the services \( A \) and \( B \) (basic activities) so that \( c^Z = c^A + c^B \), and use measures of these services, say \( q^A \) and \( q^B \), as bases for allocation of \( c^A \) and \( c^B \) respectively. However if the cost functions of the two services \( A \) and \( B \) are joint, then the total cost \( c^Z \) cannot be broken down directly into two components \( c^A \) and \( c^B \) representing separate costs of providing the two services \( A \) and \( B \). But if such a disaggregation is not done then it may become
impossible to find a suitable composite scalar measure (say $q^Z$) for the activity of the department $Z$ that would satisfy the remaining axioms, especially the symmetry and additivity axioms. The traditional approach thus yields unsatisfactory results in such cases.

We need some objective method for first apportioning the total costs to the basic activities. The Shapley Value can have an application for such cost allocation problems. Thus, as a first stage in the analysis (see figure 1), the basic activities (rather than the cost centers) can be identified as players in a $m$-person game (note that there are $i = 1, \ldots, m$ basic activities), and the total costs can be apportioned to these $m$ activities on the basis of the allocations determined by the corresponding Shapley Value axioms.

We can thus replace axiom B1, which assumes separability of the cost function, by the three Shapley Value axioms represented as in the following.

Axiom B1a  Full Cost Allocation

$$\sum_{i=1}^{n} c_i = c$$

Axiom B1b  Symmetry of Activities
If \( a_{SU(i)} = a_{SU(h)} \) for all \( S \subseteq M - \{i\} - \{h\} \)
then \( c_i = c_h \).

**Axiom B1c Additivity Over Time Periods**

If the time period \( t \) is subdivided into two time periods \( v \) and \( w \) such that \( a_S(v) + a_S(w) = a_S(t) \) for all \( S \subseteq M \), then \( c_i(v) + c_i(w) = c_i(t) \) for all \( i \in M \).

These three axioms uniquely apportion the total cost \( c \) to each of the \( i = 1, \ldots, m \) activities.

Once this apportioning has been done, we can proceed with the second stage of allocating the costs of the basic activities to the ultimate cost centers on the basis of the other three axioms in our axiom set. Note that such an application of the Shapley Value axioms would assume that the basic activities (rather than the cost centers) are uniquely identifiable. Therefore, in this case, adjoining the Shapley Value axioms to the axiom set underlying traditional cost allocation methods would not lead to an inconsistent axiom system.

**Summary and Conclusion**

Cost allocations and related cost information may serve useful purposes in providing data for better decisions or even in motivating managers of profit centers to make decisions that are congruent with the overall interests of the firm. Each such purpose may require different cost allocations or information.

In many situations, however, the problem is not how cost allocations may induce a Pareto-efficient outcome. Rather, the problem is: once a Pareto-efficient outcome has been reached, how can the total costs (or payoffs for that matter) be equitably allocated to the members of the coalition. We may not have universal agreement regarding the properties that any such allocation scheme must satisfy. Over time, however, certain methods of allocation have come to be used extensively. What are the properties of these methods that have made them generally acceptable? We have identified in this paper a set of necessary and sufficient axioms that uniquely determine a class of allocation schemes that corresponds to these traditional methods. These axioms, therefore, represent the basic and essential properties of such widely used traditional methods.

This is the spirit of what we have tried to do in identifying a simple set of basic axioms that could form a system for representing and studying traditional cost allocation methods. We also examined, with reference to this axiomatic framework, several other approaches and criteria that have been recently suggested in the accounting literature for the selection of cost allocation schemes. Thus we found that the maximum entropy criterion suggested by Lev and Theil (1978), is consistent with our axioms for traditional cost allocation methods. Further, we also found that in the case when the marginal costs are positive and diminishing, the cost allocation scheme...
determined by our axioms satisfies the core conditions discussed in the game theory literature.

The use of yet another game theoretic solution concept—the Shapley Value—for allocating costs presents interesting possibilities. In their analysis of the use of the Shapley Value for cost allocations, Roth and Verrecchia (1979) represent cost center managers as players in an n-person game and allocate costs to them on the basis of the corresponding Shapley Value axioms. However, when the Shapley Value axioms are interpreted in this manner, they are not consistent with our axioms, and hence with traditional cost allocation methods.

But traditional cost allocation methods assign costs in two stages. The total costs are first assigned to different basic activities, and then the costs for each such activity are allocated to the cost centers on the basis of some activity measure. We cannot adjoin the Shapley Value axioms to the axioms underlying traditional cost allocation methods if the resulting axiomatic system is “inconsistent”, as noted in the preceding paragraph. We, therefore, suggest an alternative interpretation of the Shapley Value axioms, representing the basic activities (rather than the cost centers) as players in a multi-person game. This interpretation insures that the two sets of axioms are “consistent”, and hence can be combined to uniquely determine a class of allocation schemes that also includes traditional allocation methods.

Our axiomatic approach thus enables us to not only integrate several recent approaches to cost allocation with a large number of commonly used allocation methods, but also to identify approaches that are not consistent with these traditional principles of cost allocation. To be sure, this axiom system was initially restricted to situations in which each common service had a separate cost function. Building on this foundation, however, we can extend the system to other more complex situations. We illustrated this possibility by reinterpreting and adjoining Shapley Value axioms to the traditional cost allocation principles, to extend the latter to situations involving joint cost functions. There are also other problems, such as idle or unused capacity, that are pertinent to the question of determining “fair” and “equitable” full cost allocation schemes. These are not dealt with in this paper. Our axiomatic approach, however, provides an interesting methodology for a systematic and logical analyses of such problems, and even incorporation of other desirable properties for cost allocation schemes into our axiom system.

In conclusion, we emphasize the fact that we have considered in this paper only the problem of determining “fair” cost allocations that are created by special contractual, statutory or organizational mechanisms. We did not examine whether these mechanisms can also insure that Pareto efficient outcomes will be attained, or whether these systems need to be restructured in order to induce efficiency. The analysis was restricted to the examination of the basic underlying properties of the traditional cost allocation methods that have evolved as acceptable solutions for the problem of
determining "equitable" allocations of full costs in many common situations. The broader task of constructing a theoretical framework to simultaneously analyze the questions of both (allocative) efficiency and (distributional) equity provides a direction for future research.

NOTES

1 We make this assumption to insure that \( c(q) \) is invertible. If the cost function is non-decreasing, but constant over some interval, then one can achieve this objective by means of small perturbations in the values of \( c(q) \) over this interval to make it strictly increasing.

2 We shall adopt here the interpretation of Shapley Value axioms suggested, for instance, by Roth and Verrecchia, representing each of \( j = 1, \ldots, n \) user departments as players in an \( n \)-person game.

3 The assumption restricting \( q \) to take up only rational number values can be relaxed. However, an additional axiom requiring "continuity" of the allocation function needs to be added to our axiom set in order to uniquely determine an allocation scheme.

4 The game corresponding to the cost allocation problem will be "convex" whenever the cost function \( c(q) \) is increasing and concave.

5 A formal axiomatic system is categorical if one and only one set of isomorphic interpretations of the formal system makes the axioms empirically true (i.e., the interpretations satisfy the axioms). Such interpretations (one or more) are called models.

6 For example, consider a very simple organization whose only costs are the salaries of two employees—one responsible for purchasing and the other for selling. The two basic activities then would be identified as purchasing and selling, and the salaries of the persons responsible for either function would be identified with the respective activities.

REFERENCES


APPENDIX

We shall demonstrate by counter-examples that no two of the three axioms A1, A2 and A3 are "sufficient" for determining a unique allocation scheme. Of course, the allocation scheme given by \( x_j = \frac{(q_j / q)}{c} \) satisfies each of the three axioms, and hence any pair of them.

To be sure one can also derive other allocation schemes from subsets of our three axioms. These will not be unique, however, and to show this we shall construct below alternative allocation schemes that satisfy both the axioms for each such pair of axioms.

1. First suppose the axiom system consists of only axioms A2 and A3. Then consider an allocation scheme given by \( x_j = \frac{(q_j / q)}{c} \), for each \( j = 1, \ldots, n \). This allocation scheme is clearly consistent with axiom A2 and A3, but not with axiom A1. Thus axioms A2 and A3 are not sufficient to uniquely determine an allocation scheme, since the ad-

junction of axiom A1 to these two axioms produces a different allocation scheme.

2. Next suppose the axiom system consists of only axioms A1 and A2. Then consider an allocation scheme given by \( x_j = c / n \) for each \( j = 1, \ldots, n \). This allocation scheme clearly satisfies axioms A1 and A2, but not axiom A3. Thus axioms A1 and A2 are not sufficient to uniquely determine an allocation scheme.

3. Finally suppose the axiom system consists of only axioms A1 and A3. Then consider an
allocation scheme given by $x_j = \gamma$, $x_j = 0$ for $j \neq 1$. This allocation scheme clearly satisfies both axioms A1 and A3, but not A2. Thus axioms A1 and A3 are not sufficient to uniquely determine an allocation scheme.

To complete our demonstration we observe that we did deduce an allocation scheme for all three axioms which was unique. No subset of these axioms uniquely determine an allocation scheme and hence we have also demonstrated that our axioms are "independent".