Modified Simes' Critical Values under Independence

Gengqian Cai, Sanat K. Sarkar[1]

GlaxoSmithKline & Temple University

Abstract

Considering independent test statistics, Simes' critical values are modified and newer sets of critical values, each providing an exact control of the type I error rate, are obtained. These modifications, as simulations show, quite often yield more powerful tests than the original Simes' test.

Keywords: Simes' test, type I error rate.

1 Introduction

Consider testing n null hypotheses $H_1, \ldots, H_n$ using right-tailed tests based on some continuous test statistics $X_1, \ldots X_n$ respectively. Assume that the X_i's are identically distributed under the null hypotheses with a common cdf $F(\cdot)$. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the ordered components of the test statistics. For testing the overall null hypothesis $H_0 = \bigcap_{i=1}^{n} H_i$ in terms of these X_i's, Simes (1986) proposed a modification of Bonferroni procedure, which can be expressed as follows:

$$\text{Reject } H_0 \text{ at level } \alpha \text{ if } X_{i:n} \geq c_i \quad \text{for any } i = 1, \ldots, n, \tag{1}$$

with c_i's satisfying $\bar{F}(c_i) = (n - i + 1)\alpha/n$, where $\bar{F}(\cdot) = 1 - F(\cdot)$. As Simes (1986) proved, it controls the type I error rate exactly at α.

Simes' test has received considerable attention by researchers in multiple hypotheses testing. Considering independent test statistics, Hochberg & Liberman (1994) have extended Simes' test allowing allotment of different weights to the different component hypotheses. Samuel-Cahn (1999) considered a generalized form of Simes' test. Krummenauer & Hommel (1999) investigated the behavior of Simes' test for discrete test statistics. Hommel (1988) employed the closure principle to extend Simes' test and developed a stepwise multiple testing procedure controlling familywise error rate (FWER). Hochberg (1988) proposed a step-up multiple testing procedure whose FWER controlling property follows from the fact that Simes' test controls its type I error rate. Hochberg & Rom (1995) discussed modifications of stepwise procedures derived from Simes' test for logically related hypotheses, leading to more powerful procedures than modified Holm's procedure in Shaffer (1986). With positively dependent test statistics, Samuel-Cahn (1996), Sarkar & Chang (1997) and Sarkar (1998) analytically verified the conservativeness of Simes' test conjectured in Simes (1986). Perhaps, the most important application of Simes's test in multiple testing is the usage of its critical values in the step-up procedure of Benjamini & Hochberg (1995) that controls false discovery rate (FDR); see also Benjamini & Yekutieli (2001) and Sarkar (2002).

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Address of correspondence: Sanat K. Sarkar, Department of Statistics, Temple University, Philadelphia, PA 19122.
email: sanat@temple.edu

While some attempts have been made recently to modify Simes' critical values when the test statistics are dependent having a multivariate normal distribution with a known common positive correlation [Kwong, Holland & Cheung (2002), Cai & Sarkar (2004)], no work is yet available on modifying these critical values under independence. In this article, we obtain newer sets of critical values, including that of the original Simes' critical values, each providing an exact control of the type I error rate. Simes' test, once modified with these newer critical values, allows more rejections of all but the most significant test statistic, thereby providing quite often a more powerful test than Simes' original test.

2 Modified Simes' Critical Values

For some fixed set of constants $c_1 \leq \cdots \leq c_n$, let us define $A_n(i), i = 1, \ldots, n, n+1$, as follows:

$$A_n(i) = \begin{cases} \Pr\{X_{1:n} \geq c_1\}, & i = 1 \\ \Pr\{X_{1:n} < c_1, \ldots, X_{i-1:n} < c_{i-1}, X_{i:n} \geq c_i\} & i = 2, \ldots, n \\ \Pr\{X_{1:n} < c_1, \ldots, X_{n:n} < c_n\} & i = n+1, \end{cases} \tag{2}$$

with the probabilities evaluated under H_0.
Note that

$$\sum_{i=1}^{n+1} A_n(i) = 1, \tag{3}$$

and, Simes' critical values are obtained by solving the equation

$$A_n(n+1) = 1 - \alpha. \tag{4}$$

First, we have the following lemma that will be used to develop one of our main theorems, Theorem 1. For a similar result, one may see Bernhard, Klein & Hommel (2004) and Finner & Roters (1994).

Lemma 1

$$A_n(i) = \frac{n}{n-i+1}\bar{F}(c_i)A_{n-1}(i), \qquad i = 1, \ldots, n.$$

Proof. Since

$$A_n(n) = \Pr\left\{X_{1:n} < c_1, \ldots, X_{n-1:n} < c_{n-1}, X_{n:n} \geq c_n\right\},$$

and the X_i's are iid, $A_n(n)$ can be expressed as

$$\begin{aligned} A_n(n) &= \binom{n}{1}\Pr\{X^{(-n)}_{1:n-1} < c_1, \ldots, X^{(-n)}_{n-1:n-1} < c_{n-1}, X_n \geq c_n\} \\ &= n\bar{F}(c_n)A_{n-1}(n), \end{aligned} \tag{5}$$

where $X^{(-n)}_{i:n-1}$ is the i^{th} ordered component of the set obtained by removing X_n from $X_1, \ldots, X_n$.

Similarly, we have

$$\begin{aligned} A_n(i) &= \Pr\{X_{1:n} < c_1, \ldots, X_{i-1:n-1} < c_{i-1}, X_{i:n} \geq c_i\} \\ &= \binom{n}{n-i+1} \Pr\{X_{1:i-1} < c_1, \ldots, X_{i-1:i-1} < c_{i-1}\} \bar{F}^{n-i+1}(c_i) \\ &= \frac{\binom{n}{n-i+1}}{\binom{n-1}{n-i}} \Pr\{X^{(-n)}_{1:n-1} < c_1, \ldots, X^{(-n)}_{i-1:n-1} < c_{i-1}, X^{(-n)}_{i:n-1} \geq c_i\} \bar{F}(c_i) \\ &= \frac{n}{n-i+1} \bar{F}(c_i) A_{n-1}(i), \end{aligned} \tag{6}$$

for $i = 1, \ldots, n-1$. When $i = n$ in (6), we get (5). This completes the proof.

Lemma 1 provides an alternative proof of Simes' result. From this lemma and eqn. (3) we see that

$$1 - A_n(n+1) = \sum_{i=1}^{n} \frac{n}{n-i+1} \bar{F}(c_i) A_{n-1}(i). \tag{7}$$

Since $\sum_{i=1}^{n} A_{n-1}(i) = 1$, it is now clear why Simes' test controls the type I error rate exactly at α. Sen (1999) gave another proof and made some remarks on Simes' test. We will obtain a more general set of c_i's satisfying (4) by deriving a more explicit expression for $A_n(n+1)$ using Lemma 1.

Note that

$$\begin{aligned} 1 - A_n(n+1) =& A_n(n) + \sum_{i=1}^{n-1} A_n(i) \\ =& n\bar{F}(c_n) A_{n-1}(n) + \sum_{i=1}^{n-1} \frac{n}{n-i+1} \bar{F}(c_i) A_{n-1}(i) \\ =& n\bar{F}(c_n) \left[1 - \sum_{i=1}^{n-1} A_{n-1}(i) \right] + \sum_{i=1}^{n-1} \frac{n}{n-i+1} \bar{F}(c_i) A_{n-1}(i) \\ =& n\bar{F}(c_n) + n \sum_{i=1}^{n-1} \left[\frac{\bar{F}(c_i)}{n-i+1} - \bar{F}(c_n) \right] A_{n-1}(i) \\ =& n\bar{F}(c_n) + n \sum_{i=1}^{n-1} \left[\frac{\bar{F}(c_i)}{n-i+1} - \bar{F}(c_n) \right] \frac{n-1}{n-i} \bar{F}(c_i) A_{n-2}(i) \\ =& n\bar{F}(c_n) + n(n-1) \sum_{i=1}^{n-1} \left[\frac{\bar{F}^2(c_i)}{(n-i+1)(n-i)} - \frac{\bar{F}(c_i)\bar{F}(c_n)}{n-i} \right] A_{n-2}(i). \end{aligned} \tag{8}$$

This provides a more explicit formula for $A_n(n+1)$.

We now solve (4) using the above expression for $A_n(n+1)$. Toward that end, we first let

$$\frac{\bar{F}^2(c_i)}{(n-i+1)(n-i)} - \frac{\bar{F}(c_i)\bar{F}(c_n)}{n-i} = \beta, \qquad i = 1, \ldots, n-1, \tag{9}$$

so that (8) reduces to

$$1 - A_n(n+1) = n\bar{F}(c_n) + n(n-1)\beta \sum_{i=1}^{n-1} A_{n-2}(i) = n\bar{F}(c_n) + n(n-1)\beta, \tag{10}$$

because of (3). We first choose c_n subject to $\bar{F}(c_n) \leq \alpha/n$, find β from (10), and put β back into (9) to solve for c_i, $i = 1, \ldots, n-1$.

Thus we have the following:

Theorem 1 *The following c_i's*

$$\bar{F}(c_i) = \frac{n-i+1}{2}\bar{F}(c_n) + \sqrt{\frac{(n-i+1)^2}{4}\bar{F}^2(c_n) + \frac{(n-i)(n-i+1)}{n(n-1)}\big(\alpha - n\bar{F}(c_n)\big)}, \tag{11}$$

$i = 1, \ldots, n$, with any c_n satisfying $\bar{F}(c_n) \le \alpha/n$, provide a solution to (4), and hence control the probability of type I error exactly at α when the test statistics are iid under the null hypotheses.

The c_i's in Theorem 1 are our proposed sets of modified Simes' critical values. Clearly, they are increasing in i. Also, by choosing $\bar{F}(c_n) = \alpha/n$, we get the original Simes' critical values. If $\bar{F}(c_n) = 0$, we get the critical values satisfying

$$\bar{F}(c_i) = \sqrt{\frac{(n-i)(n-i+1)\alpha}{n(n-1)}}, \qquad i = 1, \ldots, n.$$

The following theorem gives other useful monotonicity property of these critical values.

Theorem 2 *For each $i = 1, \ldots, n-1$, $\bar{F}(c_i)$ given in (11) is a decreasing function of $\bar{F}(c_n)$ if*

$$\alpha < \frac{n(n-i)}{(n-1)(n-i+1)} \tag{12}$$

Proof. Consider any fixed $i = 1, \ldots, n-1$, and for convenience denote $\bar{F}(c_i)$ as Y and $\bar{F}(c_n)$ as X. Also, let

$$d_1 = \frac{n-i+1}{2}, \qquad d_2 = \frac{(n-i)(n-i+1)}{n(n-1)}.$$

Then (11) can be expressed as

$$Z = \frac{Y}{X} = d_1 + \sqrt{f(X)}, \tag{13}$$

where

$$f(X) = d_1^2 + d_2\frac{\alpha - nX}{X^2}. \tag{14}$$

Taking derivative with respective to X, we have

$$\frac{dZ}{dX} = -\frac{Z}{X} + \frac{1}{X}\frac{dY}{dX} = \frac{f'(X)}{2\sqrt{f(X)}}. \tag{15}$$

Therefore, Y will be a decreasing function of X provided we have

$$-Xf'(X) > Z \cdot 2\sqrt{f(X)}, \tag{16}$$

or

$$\frac{nd_2}{X^2}\left(-X + \frac{2\alpha}{n}\right) > 2d_1\sqrt{f(X)} + 2f(X). \tag{17}$$

The inequality (17) simplifies to

$$\frac{1}{X^2}\left[\left(\frac{nd_2}{2d_1}\right)^2 - d_2\alpha\right] > 0, \tag{18}$$

that is,

$$\alpha < \frac{n(n-i)}{(n-1)(n-i+1)}. \tag{19}$$

This completes the proof.

Note that the right hand side of (12) is decreasing in i, reaching its minimum at $i = n - 1$, which is equal to $n/[2(n-1)]$. So, for any $\alpha < 1/2$, $\bar{F}(c_i)$ is a decreasing function of $\bar{F}(c_n)$ for each $i = 1, \ldots, n-1$. In other words, if we compare the modified Simes' critical values with the corresponding (unmodified) Simes' critical values, of course, when $\alpha < 1/2$, we notice that in the modified set, even though c_n is larger, all other c_i's are smaller. In other words, Simes' test once modified will allow more rejections of all but the most significant statistic. This explains why the modified Simes' critical values would quite often provide more powerful test than Simes' original test.

3 Power Comparison

In this section, we carry out simulations to investigate how does a test using modified Simes' critical values compare with the original Simes' test in terms of power. We consider $n = 10$ and $\alpha = 0.05$. We first obtain the critical values using eqn.(11) for eleven different choices of $\bar{F}(c_n)$, from 0 to $0.05/10 = 0.005$, where F is the standard normal cdf. Table 1 lists these 11 sets (in column) of $\bar{F}(c_i)$'s. Note that the last column in this table is the set of original Simes' critical values. The simulated powers using normal test statistics are reported in Table 2. For each column of powers in Table 2, the corresponding column of critical values in Table 1 was used in the simulation.

Table 1: Modified Simes' Critical Values Under Independence ($n = 10$)

i	$\bar{F}(c_i)$										
1	0.2236	0.2146	0.2051	0.1947	0.1835	0.1711	0.1572	0.1412	0.1220	0.0967	0.0500
2	0.2000	0.1920	0.1834	0.1742	0.1642	0.1531	0.1407	0.1264	0.1092	0.0867	0.0450
3	0.1764	0.1693	0.1618	0.1537	0.1449	0.1351	0.1242	0.1116	0.0965	0.0766	0.0400
4	0.1528	0.1467	0.1402	0.1332	0.1255	0.1171	0.1077	0.0968	0.0837	0.0666	0.0350
5	0.1291	0.1240	0.1185	0.1126	0.1062	0.0991	0.0911	0.0820	0.0710	0.0565	0.0300
6	0.1054	0.1013	0.0968	0.0920	0.0868	0.0810	0.0746	0.0671	0.0582	0.0464	0.0250
7	0.0816	0.0785	0.0751	0.0714	0.0674	0.0630	0.0580	0.0523	0.0454	0.0363	0.0200
8	0.0577	0.0555	0.0532	0.0506	0.0478	0.0447	0.0413	0.0373	0.0325	0.0262	0.0150
9	0.0333	0.0321	0.0308	0.0294	0.0279	0.0262	0.0243	0.0221	0.0194	0.0160	0.0100
10	0.0000	0.0005	0.0010	0.0015	0.0020	0.0025	0.0030	0.0035	0.0040	0.0045	0.0050

Note: Each column is a set of modified critical values calculated from (11) based on the corresponding $\bar{F}(c_{10})$. The last column is the original Simes' critical values.

Based on our simulation results, we see that the original Simes' test performs best compared to a modified Simes' test only in a situation when a very few of the null hypotheses are expected to be significantly false. The modified critical values quite often yield a more powerful test than the Simes' Test.

If we define $\bar{F}(c_n^*)$ to be the best $\bar{F}(c_n)$ giving the largest power, then, as our simulation indicate, $\bar{F}(c_n^*)$ is an increasing function of both the number of true null hypotheses (n_0) and the true mean for alternative hypotheses (δ).

4 Conclusion

In this article, we propose modifications to Simes' critical values when the test statistics are iid under null hypotheses. As in the original Simes' test, these modified critical values provide an exact control of the type I error rate. Simulations show that quite often these newer critical values lead to more powerful test than the original Simes' test. We should emphasize, however, that we restrict ourselves in this note only to independent test statistics and make no attempt to investigate how these modifications work when the test statistics are dependent. Moreover, there could be other ways to modify Siems' critical values.

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Table 2: Simulated Powers with Modified Simes' Critical Values Under Independence ($n = 10$ and $\alpha = 0.05$)

		$\bar{F}(c_{10})$										
n_0	δ	.0000	.0005	.0010	.0015	.0020	.0025	.0030	.0035	.0040	.0045	.0050
1	0.5	**0.258**	0.254	0.249	0.242	0.235	0.228	0.219	0.210	0.200	0.188	0.172
	1.0	**0.688**	0.681	0.671	0.659	0.643	0.626	0.606	0.581	0.551	0.510	0.445
	1.5	**0.958**	0.957	0.953	0.949	0.944	0.937	0.927	0.913	0.895	0.864	0.795
	2.0	**0.999**	0.999	0.999	0.999	0.998	0.997	0.997	0.996	0.994	0.990	0.973
	2.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999
	3.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3	0.5	**0.203**	0.201	0.198	0.194	0.189	0.184	0.178	0.172	0.164	0.157	0.146
	1.0	**0.555**	0.552	0.544	0.534	0.520	0.506	0.492	0.471	0.447	0.415	0.369
	1.5	**0.885**	0.884	0.879	0.874	0.865	0.854	0.841	0.825	0.801	0.769	0.701
	2.0	0.989	**0.990**	0.989	0.988	0.987	0.985	0.982	0.979	0.973	0.963	0.932
	2.5	1.000	**1.000**	1.000	1.000	1.000	1.000	0.999	0.999	0.999	0.998	0.995
	3.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
5	0.5	0.150	**0.151**	0.150	0.148	0.145	0.142	0.139	0.135	0.130	0.126	0.119
	1.0	0.396	**0.397**	0.394	0.388	0.381	0.372	0.363	0.350	0.334	0.315	0.287
	1.5	0.730	**0.735**	0.732	0.727	0.719	0.708	0.694	0.678	0.656	0.626	0.575
	2.0	0.936	**0.940**	0.940	0.938	0.935	0.931	0.925	0.918	0.907	0.888	0.847
	2.5	0.993	**0.994**	0.994	0.994	0.993	0.993	0.992	0.990	0.988	0.984	0.972
	3.0	1.000	1.000	1.000	1.000	**1.000**	1.000	1.000	1.000	1.000	0.999	0.998
7	0.5	0.105	**0.106**	0.106	0.105	0.104	0.103	0.102	0.100	0.097	0.094	0.091
	1.0	0.234	0.239	**0.239**	0.238	0.236	0.232	0.229	0.224	0.218	0.210	0.197
	1.5	0.458	0.476	0.479	**0.479**	0.477	0.473	0.468	0.458	0.446	0.429	0.404
	2.0	0.719	0.746	0.751	**0.752**	0.750	0.745	0.738	0.730	0.718	0.701	0.667
	2.5	0.897	0.918	0.922	**0.923**	0.922	0.921	0.919	0.914	0.908	0.897	0.873
	3.0	0.976	0.985	0.986	**0.986**	0.986	0.985	0.985	0.984	0.982	0.979	0.970
9	0.5	0.067	**0.069**	0.069	0.069	0.068	0.068	0.068	0.068	0.066	0.065	0.064
	1.0	0.097	0.103	0.105	0.106	0.106	0.107	**0.107**	0.107	0.105	0.104	0.101
	1.5	0.140	0.162	0.170	0.175	0.180	0.182	0.185	**0.186**	0.185	0.184	0.182
	2.0	0.187	0.249	0.273	0.288	0.299	0.306	0.312	0.316	0.318	**0.319**	0.318
	2.5	0.228	0.374	0.413	0.438	0.456	0.470	0.479	0.487	0.492	**0.495**	0.495
	3.0	0.256	0.525	0.576	0.608	0.629	0.646	0.658	0.668	0.674	0.680	**0.682**
10	0.0	0.051	0.052	0.052	0.052	0.052	0.052	0.052	0.052	0.051	0.051	0.050

Note: Each of these values is based on 50000 simulations involving normal test statistics. The bold font indicates the maximum power in that particular row. n_0 is the number of true null hypotheses and δ is the common mean for the alternative hypotheses. The entries in each column are the simulated powers of the the Simes' type test using the modified critical values obtained from (11) by setting $\bar{F}(c_{10})$ to the value given at the top of that column. The column with $\bar{F}(c_{10}) = 0.0050$ gives the powers of the original Simes' test.