2. Severity Distributions

Michael R. Powers, Ph.D.
Fox School, Temple University

Characterizing the Claim Amount Distributio n

Mean Residual Life Function:

For a continuous non-negative random variable X, the mean residual life at x is given by

$$E[X - x | X > x] = e(x) = \int_x^\infty (t - x) \frac{f_X(t)}{[1 - F_X(x)]} dt.$$

For discrete X, the definition is completely analogous.

For a claim amount random variable X, $e(x)$ is the expected payment per claim on a policy with a deductible of x, where claims with amounts less than or equal to x are completely ignored (i.e., not included as zeros in the average).

Empirical Mean Residual Life Function:

$$e_n(x) = \left(\frac{1}{n^*}\right)\sum_{i=1}^{n} Max\left\{(X_i - x), 0\right\},$$

where $n^* \leq n$ is the number of X_i that are greater than x.

This empirical function may be used to identify the family of distributions to which $F_X(x)$ belongs.

Limited Expected Value Function:

For a <u>continuous</u> non-negative random variable X, the limited expected value at x is given by

$$E[Min\{X,x\}] = E[X \wedge x] = \int_0^x t f_X(t)dt + x[1 - F_X(x)].$$

For <u>discrete</u> X, the definition is completely analogous.

For a claim amount random variable X, $E[X \wedge x]$ represents the expected amount per claim retained by the insured on a policy with a deductible of x.

Note that $E[X] = E[X \wedge x] + e(x)[1 - F_X(x)]$.

Empirical Limited Expected Value Function:

$$E_n\left[X \wedge x\right] = \left(\frac{1}{n}\right)\sum_{i=1}^{n} Min\left\{X_i, x\right\}.$$

This empirical function may be used to identify the family of distributions to which $F_X(x)$ belongs.

Effect of Inflation

Let $r > 0$ denote the rate of inflation for a specified period of time (e.g., a year).

For a claim amount random variable X, let $U = (1+r)X$ denote the corresponding claim amount random variable one period later.

Then

$$f_U(u) = \frac{f_X\left(\frac{u}{1+r}\right)}{1+r} \text{ for all } u, \text{ and}$$

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Note that if $F_X(x)$ is a member of the log-normal, Pareto, Burr, generalized Pareto, gamma, transformed gamma, or Weibull family of distributions, then $F_U(u)$ is a member of the same family (with different parameter values).

This is not true for the log-gamma family.

Effects of Deductibles

Truncation from Below (No Claims = Zero):

For a claim amount random variable X and deductible d, let

$$Y = \begin{cases} X \text{ if } X > d \\ \text{undefined if } X \le d \end{cases} .$$

Then

$$f_Y(y) = \begin{cases} 0 \text{ for } y \le d \\ \dfrac{f_X(y)}{1 - F_X(d)} \text{ for } y > d \end{cases} ,$$

$$F_Y(y) = \begin{cases} 0 \text{ for } y \le d \\ \dfrac{F_X(y) - F_X(d)}{1 - F_X(d)} \text{ for } y > d \end{cases} , \text{ and}$$

$$E[Y] = \int_d^\infty y \frac{f_X(y)}{1 - F_X(d)} dy = e(d) + d .$$

Censoring from Below (Small Claims = Zero):

For a claim amount random variable X and deductible d, let

$$Y = \begin{cases} X \text{ if } X > d \\ 0 \text{ if } X \leq d \end{cases}.$$

Then

$$f_Y(y) = \begin{cases} F_X(d) \text{ for } y = 0 \\ 0 \text{ for } 0 < y \leq d \\ f_X(y) \text{ for } y > d \end{cases},$$

$$F_Y(y) = \begin{cases} F_X(d) \text{ for } y = 0 \\ F_X(d) \text{ for } 0 < y \leq d \\ F_X(y) \text{ for } y > d \end{cases}, \text{ and}$$

$$E[Y] = (0)F_X(d) + \int_d^\infty y f_X(y)\,dy = [e(d) + d][1 - F_X(d)] .$$

Truncation from Below, with Shifting (No Claims = Zero):

For a claim amount random variable X and deductible d, let

$$W = \begin{cases} X - d \text{ if } X > d \\ \text{undefined if } X \le d \end{cases}.$$

Then

$$f_W(w) = \frac{f_X(w+d)}{1 - F_X(d)} \text{ for } w > 0,$$

$$F_W(w) = \frac{F_X(w+d) - F_X(d)}{1 - F_X(d)} \text{ for } w > 0, \text{ and}$$

$$E[W] = \int_0^\infty w \frac{f_X(w+d)}{1 - F_X(d)} dw = e(d) \; .$$

Censoring from Below, with Shifting (Small Claims = Zero):

For a claim amount random variable X and deductible d, let

$$W = \begin{cases} X - d \text{ if } X > d \\ 0 \text{ if } X \le d \end{cases} .$$

Then

$$f_W(w) = \begin{cases} F_X(d) \text{ for } w = 0 \\ f_X(w+d) \text{ for } w > 0 \end{cases} ,$$

$$F_W(w) = \begin{cases} F_X(d) \text{ for } w = 0 \\ F_X(w+d) \text{ for } w > 0 \end{cases} , \text{ and}$$

$$E[W] = (0)F_X(d) + \int_0^\infty w f_X(w+d)\,dw = e(d)\left[1 - F_X(d)\right] .$$

Loss Elimination Ratio:

For any policy provision, such as a deductible, limit, or other coverage restriction, the loss elimination ratio (LER) denotes the proportion of total claim costs that is eliminated by the provision; i.e.,

$$LER = 1 - \frac{\text{Residual Claim Costs}}{\text{Total Claim Costs}}.$$

For a claim amount random variable X with deductible d,

$$LER = 1 - \frac{e(d)\left[1 - F_X(d)\right]}{E\left[X\right]} = \frac{E\left[X \wedge d\right]}{E\left[X\right]} \quad .$$

Effects of Policy Limits

Truncation from Above (No Claims = Policy Limit):

For a claim amount random variable X and policy limit ℓ, let

$$Z = \begin{cases} X \text{ if } X < \ell \\ \text{undefined if } X \geq \ell \end{cases} .$$

Then

$$f_Z(z) = \begin{cases} \dfrac{f_X(z)}{F_X(\ell)} \text{ for } z < \ell \\ 0 \text{ for } z \geq \ell \end{cases} ,$$

$$F_Z(z) = \begin{cases} \dfrac{F_X(z)}{F_X(\ell)} \text{ for } z < \ell \\ 1 \text{ for } z \geq \ell \end{cases} , \text{ and}$$

$$E[Z] = \int_0^{\ell} z \frac{f_X(z)}{F_X(\ell)} dz = \frac{E[X \wedge \ell] - \ell[1 - F_X(\ell)]}{F_X(\ell)} .$$

Censoring from Above (Large Claims = Policy Limit):

For a claim amount random variable X and policy limit ℓ, let

$$Z = \begin{cases} X \text{ if } X < \ell \\ \ell \text{ if } X \geq \ell \end{cases} .$$

Then

$$f_Z(z) = \begin{cases} f_X(z) \text{ for } z < \ell \\ 1 - F_X(\ell) \text{ for } z = \ell \\ 0 \text{ for } z > \ell \end{cases} ,$$

$$F_Z(z) = \begin{cases} F_X(z) \text{ for } z < \ell \\ 1 \text{ for } z = \ell \\ 1 \text{ for } z > \ell \end{cases} , \text{ and}$$

$$E[Z] = \int_0^{\ell} z f_X(z)dz + \ell\left[1 - F_X(\ell)\right] = E[X \wedge \ell] .$$

Loss Elimination Ratio:

For a claim amount random variable X with policy limit ℓ,

$$LER = 1 - \frac{E[X \wedge \ell]}{E[X]} = \frac{e(\ell)[1 - F_X(\ell)]}{E[X]} \quad .$$