Bayesian vs. Bühlmann Credibility

- *Result 1:*

  The Bühlmann credibility estimator is always the best linear least-squares approximation to the Bayesian credibility estimator.

  That is, among all linear functions $a\overline{X} + b$, the selection

  $$C = a^* \overline{X} + b^*$$

  minimizes the expression $E_{\mathbf{X}}\left[\left(E_{\theta|\mathbf{X}}\left[\mu(\theta) \mid X_1, X_2, \ldots, X_n\right] - a\overline{X} - b\right)^2\right]$.

  For a discrete model with possible Bayesian credibility estimates $B_1, B_2, \ldots, B_m$ and possible sample means $R_1, R_2, \ldots, R_m$, the Bühlmann credibility estimator must be the linear function of R_i,

  $$C_i = aR_i + b,$$

  that minimizes the expression $\sum_{i=1}^{m} w_i\left(B_i - aR_i - b\right)^2$, where $w_i = \Pr\left\{\overline{X} = R_i\right\}$.

- *Result 2:*

  If, given θ, the X_i are *i.i.d.* random variables from a linear exponential family, and the prior distribution of θ is conjugate to this family, then the Bühlmann credibility estimator is equivalent to the Bayesian credibility estimator.

  Some cases where Bühlmann credibility $\Leftrightarrow$ Bayesian credibility:

  beta-binomial, gamma-Poisson, normal-normal, and inverted gamma-exponential conjugate families.

- *Example I* (gamma-Poisson conjugate family):

  Let $X_i \mid \lambda \sim$ i.i.d. Poisson(λ), where $\lambda \sim$ gamma(α, β).

  Then $\lambda \mid X_1, X_2, \ldots, X_n \sim \text{gamma}\left(\alpha + \sum_{i=1}^{n} X_i, \beta + n\right)$, and so the Bayesian posterior mean is

  $$E_{\theta|\mathbf{X}}\left[\mu(\theta) \mid X_1, X_2, \ldots, X_n\right] = \frac{\alpha + \sum_{i=1}^{n} X_i}{\beta + n}.$$

The Bühlmann credibility estimator is

$$C = z\overline{X} + (1-z)\frac{\alpha}{\beta}, \text{ where}$$

$$z = \frac{n}{n + \dfrac{EVPV}{VHM}} = \frac{n}{n + \dfrac{E_\lambda[\lambda]}{Var_\lambda[\lambda]}} = \frac{n}{n + \dfrac{\alpha/\beta}{\alpha/\beta^2}} = \frac{n}{n+\beta}.$$

Thus,

$$C = \frac{n}{n+\beta}\overline{X} + \frac{\beta}{n+\beta}\left(\frac{\alpha}{\beta}\right) = \frac{\sum_{i=1}^{n} X_i}{n+\beta} + \frac{\alpha}{n+\beta} = \text{the posterior mean.}$$

Example II (beta-Bernoulli conjugate family):

Let $X_i | p \sim$ i.i.d. Bernoulli(p), where $p \sim$ beta(a,b).

Then $p | X_1, X_2, \ldots, X_n \sim$ beta$\left(a + \sum_{i=1}^{n} X_i, b + n - \sum_{i=1}^{n} X_i\right)$, and so the Bayesian posterior mean is

$$E_{\theta|\mathbf{x}}\left[\mu(\theta) | X_1, X_2, \ldots, X_n\right] = \frac{a + \sum_{i=1}^{n} X_i}{a+b+n}.$$

The Bühlmann credibility estimator is

$$C = z\overline{X} + (1-z)\frac{a}{a+b}, \text{ where}$$

$$z = \frac{n}{n + \dfrac{EVPV}{VHM}} = \frac{n}{n + \dfrac{E_p[p(1-p)]}{Var_p[p]}}$$

$$= \frac{n}{n - 1 + \dfrac{[a/(a+b)] - [a/(a+b)]^2}{ab/(a+b+1)(a+b)^2}} = \frac{n}{n - 1 + (a+b+1)} = \frac{n}{n+a+b}.$$

Thus,

$$C = \frac{n}{n+a+b}\overline{X} + \frac{a+b}{n+a+b}\left(\frac{a}{a+b}\right) = \frac{\sum_{i=1}^{n} X_i}{n+a+b} + \frac{a}{n+a+b} = \text{the posterior mean.}$$

- Pure premium calculations:

  If B is the Bayesian credibility estimator of expected pure premium, then an alternative approach is to calculate the Bayesian estimators of expected frequency, F, and expected severity, S, and use the product, $F \times S = B$.

  If C is the Bühlmann credibility estimator of expected pure premium, and F and S are the Bühlmann estimators of expected frequency and expected severity, respectively, then the alternative estimator $F \times S$ will equal C only in special cases.

- Some further observations:

  (1) z is an increasing function of n, and, for fixed values of $EVPV$ and VHM,

  $$z \to 1 \text{ as } n \to +\infty.$$

  (2) z is a decreasing function of $EVPV$, and, for fixed values of n and VHM,

  $$z \to 0 \text{ as } EVPV \to +\infty, \text{ and } z \to 1 \text{ as } EVPV \to 0.$$

  (3) z is an increasing function of VHM, and, for fixed values of n and $EVPV$,

  $$z \to 1 \text{ as } VHM \to +\infty, \text{ and } z \to 0 \text{ as } VHM \to 0$$

  (Thus, a large value of z is not necessarily a good thing.)

  (4) Although the Bühlmann credibility estimator C must fall between H and R, there is no such restriction on the Bayesian credibility estimator B <u>if it is different from</u> C.

  Specifically, if the parameter space, Θ, is discrete, then

  $$VHM \to +\infty \quad \begin{cases} C \to R \\ B \to \theta \in \Theta, \text{ for the closest } \theta \text{ to } R \end{cases}.$$

Note that the first three of the above results can be shown rigorously by taking the appropriate partial derivatives of $z = \dfrac{n}{n+k}$, where $k = \dfrac{EVPV}{VHM}$.