Chapter 6. Inner Product Spaces

- So far our primary use of vector spaces and subspaces has been as algebraic structures.

- For instance, the algebra of linear combinations of vectors was used to generate solutions of a homogeneous linear system **Ax = 0**. We also used bases of eigenvectors to develop approximations and compress information contained in a matrix.

- However, using the dot product in R^n we saw that we could determine the length of a vector and, at least for R^2 and R^3, we developed the notion of an angle between vectors. These concepts hinted at geometric aspects of vector spaces and subspaces.

Section 6.1 The Dot Product in R^n and C^n

The dot product of a pair of vectors **x** and **y** in R^n is

$$\mathbf{x} \bullet \mathbf{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \bullet \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$$
$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{j=1}^{n} x_j y_j \quad .$$

We refer to (1) as the **standard dot product** on R^n.

The complex dot product of a pair of vectors **x** and **y** in C^n is

$$\overline{\mathbf{x}} \bullet \mathbf{y} = \sum_{j=1}^{n} \overline{x_j}\, y_j$$

where the bar indicates the conjugate.

It is convenient at this time to introduce a notation for dot products that can be used for R^n or C^n, where we let the nature of the vectors determine whether to use the computational steps in expressions.

For vectors x and y let (x,y) denote the dot product.

Example 1.

a) Let $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 5 \\ 8 \end{bmatrix}$. Then from (1) we have

$$\mathbf{(x,y)} = (3)(2) + (2)(-2) + (-1)(5) + (0)(8) = 6 - 4 - 5 + 0 = -3.$$

b) Let $\mathbf{v} = \begin{bmatrix} 2-i \\ 2i \\ 4+3i \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1+2i \\ 3-2i \\ 2 \end{bmatrix}$. Then from (2) we have

$$\mathbf{(v,w)} = \left(\overline{2-i}\right)(1+2i) + \left(\overline{2i}\right)(3-2i) + \left(\overline{4+3i}\right)(2) = 4-7i.$$

The **length** of a vector **x** in R^n or C^n is denoted by $\|\mathbf{x}\|$ and is determined from the expression

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x},\mathbf{x})} = \sqrt{\sum_{j=1}^{n} |x_j|^2}\,.$$

We say that a pair of vectors **x** and **y** are **orthogonal** provided **(x,y)** = 0.

IMPORTANT RESULT:

Show that the set of all vectors orthogonal to a vector v is a subspace.

The dot product gives us a tool with which we can define the distance between vectors. For a pair of vectors **x** and **y** in R^n or C^n, the **distance** between the two vectors is given by

$$\mathbf{D(x,y)} = \|\mathbf{x} - \mathbf{y}\|.$$

<u>Example</u> Let $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 5 \\ 8 \end{bmatrix}$. Then

$$\mathbf{D(x,y)} = \|\mathbf{x} - \mathbf{y}\| = \left\| \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 5 \\ 8 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ 4 \\ -6 \\ -8 \end{bmatrix} \right\| = \sqrt{101}.$$

<u>Terminology</u>

Dot product in R^n and C^n.	The length or norm of a vector.
Unit vector.	Orthogonal vectors.
The distance between vectors.	