6.1

Functions and Equations in Two Variables

- Evaluate functions of two variables
- Understand basic concepts about systems of equations in two variables
- Apply the method of substitution

Introduction

Many quantities in everyday life depend on more than one variable.

Examples:

Area of a rectangle requires both width and length.

Heat index is the function of temperature and humidity.

Wind chill is based on the temperature and wind speed.

Grade point average is computed using grades and credit hours.

Functions in Two Variables

- **The arithmetic operations of addition, subtraction, multiplication, and division are computed by *functions of two inputs*.**

- **For example, the addition function f can be represented symbolically by $f(x,y) = x + y$, and we can also name output like $z = f(x,y)$.**

- **The *independent variables* are x and y.**

- **The *dependent variable* is z. The z output depends on the values of inputs x and y.**

Example

For each function, evaluate the expression and interpret the result.

a) $f(5, -2)$ where $f(x,y) = xy$

b) $A(6,9)$, where $A(b,h) = \frac{1}{2} bh$ calculates the area of a triangle with a base of 6 inches and a height of 9 inches.

Solution

a) $f(5, -2) = (5)(-2) = -10$. The product of 5 and -2.

b) $A(6,9) = \frac{1}{2}(6)(9) = 27$

If a triangle has a base of 6 inches and a height of 9 inches, then the area of the triangle is 27 square inches.

Example:

The equation $V = LWH$ gives the volume of a rectangular box where L is the length of the base, W is the width of the base, and H is the height of the box.

a) Solve $V = LWH$ for L.

b) Find L when W = 6.5 ft, H = 9 ft, and $V = 187.2\ \text{ft}^3$.

Solution

a)

$$V = LWH$$

$$\frac{V}{WH} = L$$

b)

$$\frac{V}{WH} = L$$

$$\frac{187.2}{(6.5)(9)} = L$$

$$3.2\ \text{ft} = L$$

Colors on Computer Monitors

Colored light is a combination of intensities of red, green, and blue light. Many computer monitors are capable of creating 256 intensities (numbered 0 through 255) for each of these three colors. Zero indicates the absence of color, while 255 represents te brightest intensity of a color. A color displayed on a screen is determined by a function f(r, g, b). So the color you see is a function of 3 variables.

We can illustrate this in MATLAB using routine

rgbexamp

How many colors can be obtained? 16,777,216

Systems of Equations

- A ***linear equation*** *in two variables* can be written in the form $ax + by = k$, where *a*, *b*, and *k* are constants.
- A pair of equations is called a ***system of linear equations*** because they involve solving more than one linear equation.

$$ax + by = k$$
$$cx + dy = h$$

  - A ***solution*** to this system of equations consists of an *x*-value and a *y*-value that satisfy both equations simultaneously.
  - The set of all solutions is called the ***solution set***.
  - A system of linear equations can have more than 2 equations.

From Page 479 in the text.

THE METHOD OF SUBSTITUTION

To use the method of substitution to solve a system of two equations in two variables, perform the following steps.

STEP 1: Choose a variable in one of the two equations. Solve the equation for that variable.

STEP 2: Substitute the result from **STEP 1** into the other equation and solve for the remaining variable.

STEP 3: Use the value of the variable from **STEP 2** to determine the value of the other variable. To do this, you may want to use the equation you found in **STEP 1**.

Note: To check your answer, substitute the value of each variable into the *given* equations. These values should satisfy *both* equations.

We illustrate the method of substitution by examples.

Example:

Solve the system.

$$4\mathbf{x} + 2\mathbf{y} = 8$$
$$3\mathbf{x} - 7\mathbf{y} = -11$$

Solution

Step 1: Solve one of the equations for one of the variables. Here we start with equation 1 and solve for y.

$$4\mathbf{x} + 2\mathbf{y} = 8$$
$$2\mathbf{y} = -4\mathbf{x} + 8$$
$$\mathbf{y} = -2\mathbf{x} + 4$$

Step 2: Substitute -2**x** + 4 for ***y*** in the second equation and solve for x.

$$3\mathbf{x} - 7\left(-2\mathbf{x} + 4\right) = -11$$
$$3\mathbf{x} + 14\mathbf{x} - 28 = -11$$
$$17\mathbf{x} = 17$$
$$\mathbf{x} = 1$$

Solution (continued)

Step 3: Substitute $x = 1$ into the equation $\mathbf{y} = -2\mathbf{x} + 4$ from **Step 1**. We find that

$$\mathbf{y} = 2$$

Check: Substitute x = 1 and y = 2 into each of the original equations.

$$4\mathbf{x} + 2\mathbf{y} = 8 \qquad 3\mathbf{x} - 7\mathbf{y} = -11$$

$$4(1) + 2(2) = 8$$

$$3(1) - 7(2) = -11$$

The ordered pair (1, 2) is the solution to the system since it “checks” in *both* equations.

Example:

Solve the system.

$$8\mathbf{x} - 2\mathbf{y} = -4$$
$$-4\mathbf{x} + \mathbf{y} = 2$$

Solution

Solve the second equation for *y*.

$$-4\mathbf{x} + \mathbf{y} = 2$$
$$\mathbf{y} = 4\mathbf{x} + 2$$

Substitute 4*x* + 2 for *y* in the first equation, solving for *x*.

$$8\mathbf{x} - 2(4\mathbf{x} + 2) = -4$$
$$8\mathbf{x} - 8\mathbf{x} - 4 = -4$$
$$-4 = -4$$

The equation –4 = –4 is an identity that is always true and indicates that there are **infinitely many solutions**. The two equations are equivalent; that is, they are the same line. The equations are multiples of one another. Here -2 times equation 2 gives equation 1. (If you graphed the equations they would fall on top of one another.)

Example:

Solve the system.

$$0.5\mathbf{x} - 0.4\mathbf{y} = 2$$
$$-2\mathbf{x} + 1.6\mathbf{y} = 4$$

Solution

Solve the first equation for *y*.

$$0.5\mathbf{x} - 0.4\mathbf{y} = 2$$
$$-0.4\mathbf{y} = 2 - 0.5\mathbf{x}$$
$$\mathbf{y} = \frac{2 - 0.5\mathbf{x}}{-0.4} = -50 + 1.25\mathbf{x}$$

Substitute -50 + 1.25x for *y* in the second equation, then solving for *x*.

$$-2\mathbf{x} + 1.6\left(-50 + 1.25\mathbf{x}\right) = 4$$
$$-2\mathbf{x} - 80 = 2\mathbf{x} = 4$$
$$-80 = 4$$

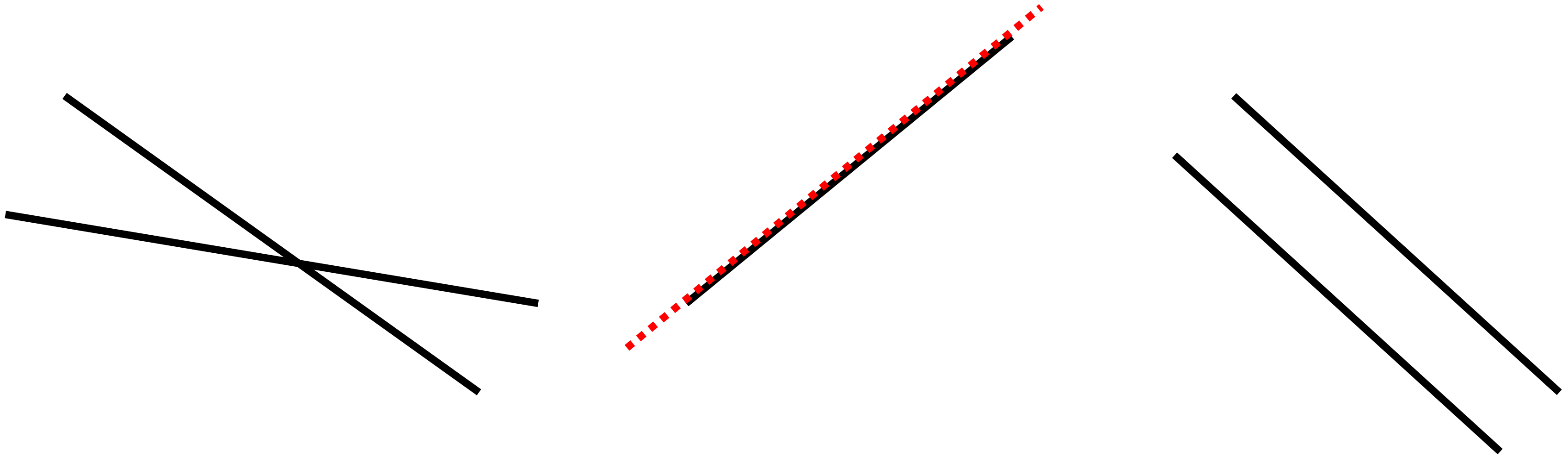
🡐 Contradicts a known fact, so there is no solution to the linear system. So the two lines do not intersect.

Possible solution sets for a system of two equations in two unknowns:

A unique solution; exactly one ordered pair (r, s) that satisfies both equations. The graphs of the corresponding lines intersect *once*.

Infinitely many solutions; the graphs of the corresponding lines are identical.

No solutions; no ordered pair (r, s) satisfies both equations. The graphs of the corresponding lines do not intersect.



Example: During the first four months of 2004, a total of 6767 people were vaccinated for smallpox in Florida and Texas. There were 273 more people vaccinated in Florida than in Texas.

Determine a system of linear equations whose solution represents the number of vaccinations given in each state.

Let x = the number of vaccinations in Florida

and y = the number of vaccinations in Texas

Determine equations in x and y that are derived from the problem statement.

During the first four months of 2004, a total of 6767 people were vaccinated for smallpox in Florida and Texas.

$$x + y = 6767$$

There were 273 more people vaccinated in Florida than in Texas.

$$x = 273 + y$$

The system is $\begin{aligned} x + y &= 6767 \\ x - y &= 273 \end{aligned}$ **Solve this system.**

$y = 6767 - x \Rightarrow x - (6767 - x) = 273 \Rightarrow 2x - 6767 = 273$

$\Rightarrow 2x = 7040 \quad \Rightarrow x = 3520$

$\Rightarrow y = 6767 - x = 6767 - 3520 = 3247$

Example: Libraries have moved to online catalogs for holdings. From 1968 to 1996 the number of 3 × 5-inch cards sold to libraries by the Library of Congress declined by 78.19 million. The number of cards sold in 1996 was only 0.72% of the 1968 number. How many cards were sold in 1968 and in 1996? Round your answers to the nearest hundredth of a million.

Let x = the number of cards sold in 1968 — Set up equations.

and y = the number of cards sold in 1996.

From 1968 to 1996 the number of 3 × 5-inch cards sold to libraries by the Library of Congress declined by 78.19 million.

$$y = x - 78.19$$

The number of cards sold in 1996 was only 0.72% of the 1968 number.

$$y = 0.0072x$$

Solve the system

$$y = x - 78.19$$
$$y = 0.0072x$$

➔ $0.0072x = x - 78.19$

➔ $x \approx 78.75705\ldots \approx 78.76$

Then $y \approx 0.0072(78.76) \approx 0.567072\ldots \approx 0.57$

Example: **Find the intersection point between line L and line Q given that**

L: 2x + 4y = 10

Q: is the line perpendicular to L that goes through point (0, 5)

First find the equation of line Q, then use the method of substitution.

Using line L we have 2x + 4y = 10, which we solve for y, obtaining

$$4y = -2x + 10 \qquad \Rightarrow y = \frac{-2}{4}x + \frac{10}{4}$$

So the slope of line L is -1/2 which makes the slope of line Q equal to 2. Thus line Q have equation

$$y = 2x + 5$$

Next solve the linear system

$$2x + 4y = 10$$

$$y = 2x + 5$$

Substitute $y = 2x + 5$ into $2x + 4y = 10$ and solve for x. We get

$$2x + 4(2x + 5) = 10$$

$$2x + 8x + 20 = 10$$

$$10x + 20 = 10 \rightarrow 10x = -10 \rightarrow x = -1$$

Then in $y = 2x + 5$ set $x = 1$, obtaining $y = 3$.

So the two lines intersect in the point (-1, 3).

Use the method of substitution to solve each of the following linear systems.

$$3x - 2y = 16$$
$$4x + 3y = 10$$

$$\frac{1}{2}x - \frac{3}{4}y = \frac{1}{2}$$
$$\frac{1}{5}x - \frac{3}{10}y = \frac{1}{5}$$

$$0.6x - 0.2y = 2$$
$$-1.2x + 0.4y = 3$$

$$100x + 200y = 300$$
$$200x + 100y = 0$$

Use the method of substitution to solve each of the following linear systems.

$$3x - 2y = 16$$
$$4x + 3y = 10$$

x = 4, y = -2

$$\frac{1}{2}x - \frac{3}{4}y = \frac{1}{2}$$
$$\frac{1}{5}x - \frac{3}{10}y = \frac{1}{5}$$

Infinitely many solutions.

$$0.6x - 0.2y = 2$$
$$-1.2x + 0.4y = 3$$

No solutions; solution set is empty.

$$100x + 200y = 300$$
$$200x + 100y = 0$$

x = -1, y = 2