5.6

Exponential and Logarithmic Equations

Solve exponential equations

Solve logarithmic equations

We will be using these properties of logarithms.

Common Logs

$$\log(1) = 0$$

$$\log(mn) = \log(m) + \log(n)$$

$$\log\left(\frac{m}{n}\right) = \log(m) - \log(n)$$

$$\log(m^r) = r\log(m)$$

Natural Logs

$$\ln(1) = 0$$

$$\ln(mn) = \ln(m) + \ln(n)$$

$$\ln\left(\frac{m}{n}\right) = \ln(m) - \ln(n)$$

$$\ln(m^r) = r\ln(m)$$

Modeling Compound Interest

How long does it take $100 to "double" if invested into an account which compounds quarterly at an annual rate of 5%?

You need to solve for *t* in the following equation

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substituting the data from the problem we need to solve for t in

$$200 = 100\left(1 + \frac{.05}{4}\right)^{4t}$$

Solve for *t*

$$200 = 100\left(1+\frac{.05}{4}\right)^{4t}$$

$$2 = (1.0125)^{4t}$$

Divide each side by 100

$$\log 2 = \log(1.0125)^{4t}$$

Take common logarithm of each side

$$\log 2 = 4t\log 1.0125$$

Property: log(m^r) = *r*log*m*

$$4t\log 1.0125 = \log 2$$

$$t = \frac{\log 2}{4\log 1.0125}$$

Divide each side by 4log1.0125

$$t \approx 13.95\,\text{years}$$

Approximate using calculator

Alternatively

$$200 = 100\left(1 + \frac{.05}{4}\right)^{4t}$$

$$2 = (1.0125)^{4t}$$

Divide each side by 100

$$\ln 2 = \ln(1.0125)^{4t}$$

Take natural logarithm of each side

$$\ln 2 = 4t \ln 1.0125$$

Property: ln(m^r) = r ln$m$

$$4t \ln 1.0125 = \ln 2$$

$$t = \frac{\ln 2}{4 \ln 1.0125}$$

Divide each side by 4ln1.0125

$$t \approx 13.95 \text{ years}$$

Approximate using calculator

Note that the only difference in this and the previous solution is that we took natural logs of both sides of the equation instead of common logs.

Follow up question.

How long does it take $10,000 to “double” if invested into an account which compounds quarterly at an annual rate of 5%?

Solve $3(1.2)^x + 2 = 15$ for x symbolically

$$3(1.2)^x = 13$$

$$1.2^x = \frac{13}{3}$$

Divide each side by 3

$$\log 1.2^x = \log\left(\frac{13}{3}\right)$$

Take common logarithm of each side
(Could use natural logarithm)

$$x \log 1.2 = \log\left(\frac{13}{3}\right)$$

Property: $\log(m^r) = r \log m$

$$x = \frac{\log\left(\frac{13}{3}\right)}{\log 1.2}$$

Divide each side by log1.2

$$x \approx 8.04$$

Approximate using calculator

Solve $e^{x+2} = 5^{2x}$ for x symbolically

$$e^{x+2} = 5^{2x}$$

$$\ln\left(e^{x+2}\right) = \ln\left(5^{2x}\right)$$ **Take natural logarithm of each side**

$$(x+2)\ln e = 2x\ln 5$$ **Property: ln(m^r) = r ln$m$**

$$x+2 = 2x\ln 5$$ **lne = 1**

$$x - 2x\ln 5 = -2$$ **Subtract 2$x$ln5 and 2 from each side**

$$x(1-2\ln 5) = -2$$ **Factor x from left-hand side**

$$x = \frac{-2}{1-2\ln 5}$$ **Divide each side by 1 – 2ln5**

$$x \approx .901$$ **Approximate using calculator**

Modeling World Population

The population P of the world was 3 billion in 1960, 6 billion in1999, and can be modeled by

$$P(x) = 3(1.018)^{x-1960}$$

where x is the year.

(a) Determine the year in which the population will reach 8 billion.

(b) By what percentage does the world population grow, on average, each year using this model?

$$P(x) = 3(1.018)^{x-1960}$$

(a) Determine the year in which the population will reach 8 billion.

Set P = 8 and solve for x.

$$8 = 3(1.018)^{x-1960}$$

$$\frac{8}{3} = (1.018)^{x-1960}$$

$$\log\left(\frac{8}{3}\right) = \log\left((1.018)^{x-1960}\right)$$

$$= (x-1960)\log(1.018)$$

Then we have

$$\log\left(\frac{8}{3}\right) = (x-1960)\log(1.018)$$

$$x - 1960 = \frac{\log\left(\frac{8}{3}\right)}{\log(1.018)}$$

$$x = 1960 + \frac{\log\left(\frac{8}{3}\right)}{\log(1.018)} \approx 1960 + 54.97 \approx 2014.97$$

So about 2015.

$$P(x) = 3(1.018)^{x-1960}$$

(b) By what percentage does the world population grow, on average, each year using this model?

This is an exponential growth model with base $a = 1.018 = 1 + 0.018$ so the annual percent of change is 1.8%.

Solving a Logarithmic Equation Symbolically

In developing countries there is a relationship between the amount of land a person owns and the average daily calories consumed. This relationship is modeled by the formula $C(x) = 280 \ln(x+1) + 1925$ where x is the amount of land owned in acres and

$$0 \le x \le 5$$

(Source: D. Gregg: *The World Food Problem*)

Determine the number of acres owned by someone whose average intake is 2400 calories per day.

Must solve for *x* in the equation

$$280 \ln(x+1) + 1925 = 2400$$

Solve 280 ln(*x*+1) + 1925 = 2400 Symbolically

$$280\ln(x+1)+1925=2400$$

$$280\ln(x+1)=2400-1925$$

Subtract 1925 from each side

$$280\ln(x+1)=475$$

$$\ln(x+1)=\frac{475}{280}$$

Divide each side by 280

$$e^{\ln(x+1)}=e^{\frac{475}{280}}$$

Exponentiate each side base *e*

$$x+1=e^{\frac{475}{280}}$$

Inverse property $e^{\ln k}=k$

$$x=e^{\frac{475}{280}}-1$$

Subtract 1 from each side

$$x\approx 4.45$$

Approximate using calculator

Find a modeling function for the number of people waiting for organ transplants in a given year.

The gap between available organs for transplants and people who need them has widened. In 1999 about 60,000 people were waiting for organ transplants and in 2002 this number increased to 80,000. This trend is modeled by the equation $T(x) = Ca^{x-1999}$ where C and a are constants and x is a year ≥ 1999.

Find C and a and let T be in 1,000s of people. Then estimate the number of people waiting for organ transplants in 2006.

$$T(x) = Ca^{x-1999}$$

Known data: x = 1999, T = 60 and x = 2002, T = 80

Using x = 1999 and T = 60 in the equation we get ➔
$60 = T(1999) = Ca^{1999-1999}$ so $C = 60$

Next using $T(x) = 60\ a^{x-1999}$ with data x = 2002 and T = 80 we have

$$80 = T(2002) = 60\ a^{2002-1999} = 60\ a^3$$

Solving for a in $60\ a^3 = 80$ we get

$$a^3 = \frac{80}{60}$$

$$a = \sqrt[3]{\frac{4}{3}} \approx 1.10$$

So $T(x) = 60\ (1.1)^{x-1999}$

Then $T(2006) \approx 117$.

Credit Cards

From 1987 to 1996 the number of Visa cards and MasterCards was up 80% to 376 million. The function

$$f(x) = 36.2e^{0.14x}$$

models the amount of credit card spending from Thanksgiving to Christmas in billions of dollars. In this formula $x = 0$ corresponds to 1987 and $x = 9$ to 1996. (Source: National Credit Counseling Services.)

Determine the year when this amount reached $55 billion.

Credit Cards $f(x) = 36.2e^{0.14x}$

x = 0 corresponds to 1987 and x = 9 to 1996

Determine the year when this amount reached $55 billion.

$$f(x) = 36.2e^{0.14x}$$ Set f(x) = 55 and solve for x.

$$55 = 36.2e^{0.14x}$$ Divide by 36.2

$$\frac{55}{36.2} = e^{0.14x}$$ Take the ln of both sides.

$$\ln\left(\frac{55}{36.2}\right) = \ln\left(e^{0.14x}\right)$$ Use inverse function properties.

$$\ln\left(\frac{55}{36.2}\right) = 0.14x$$ Solve for x.

$$\frac{1}{0.14}\ln\left(\frac{55}{36.2}\right) = x \Rightarrow x = 2.99 \Rightarrow x = 3 \Rightarrow 1990$$

Solving Exponential Equations

Solve each equation.

(a) $5(1.2)^x + 1 = 26$

(b) $5^{x-3} = e^{2x}$

Solving Exponential Equations

Solve

$$5(1.2)^x + 1 = 26$$

Subtract 1 from both sides

$$5(1.2)^x = 25$$

Divide by 5

$$(1.2)^x = 5$$

Take the log of both sides

$$\log(1.2)^x = \log(5)$$

Use properties of logs

$$x\log(1.2) = \log(5)$$

Solve for x

$$x = \frac{\log(5)}{\log(1.2)} \Rightarrow x \approx 8.827$$

Solving Exponential Equations

Solve

$$5^{x-3} = e^{2x}$$

Take the ln of both sides

$$\ln(5^{x-3}) = \ln(e^{2x})$$

Use properties & logs & inverse functions

$$(x-3)\ln(5) = 2x$$

Expand & collect terms

$$x\ln(5) - 3\ln(5) = 2x$$

$$x\ln(5) - 2x = 3\ln(5)$$

$$(\ln(5) - 2)x = 3\ln(5)$$

Solve for x

$$x = \frac{3\ln(5)}{\ln(5) - 2} \Rightarrow x \approx -12.36$$

Solving Logarithmic Equations

Solve each equation.

(a) $2\ln(x+1) = \ln(1-2x)$

(b) $\log(x^5) = 4 + 3\log(x)$

Solving Logarithmic Equations

Solve $2\ln(x+1) = \ln(1-2x)$ — Use properties of logarithms.

$\ln(x+1)^2 = \ln(1-2x)$ — Both sides as exponents on e.

$e^{\ln(x+1)^2} = e^{\ln(1-2x)}$ — Use properties of inverse functions.

$(x+1)^2 = (1-2x)$ — Expand the left side.

$x^2 + 2x + 1 = (1-2x)$ — Collect like terms & simplify.

$x^2 + 4x = 0$ — Solve for x.

$x = 0 \text{ or } x = -4$ — Check in the original equation.

X = 0 ➔

$2\ln(0+1) = \ln(1-0)$

$2\ln(1) = \ln(1)$, but $\ln(1) = 0$

so this is true and $x = 0$ is a solution

x = - 4 ➔

$2\ln(-4+1) = \ln(1-2(-4))$

$2\ln(-3) = \ln(9)$

Why is -4 not a solution?

Solving Logarithmic Equations

Solve $\log(x^5) = 4 + 3\log(x)$

Use properties of logs.

$5\log(x) = 4 + 3\log(x)$

Group common terms & simplify

$5\log(x) - 3\log(x) = 4$

$2\log(x) = 4$

Divide by 2.

$\log(x) = 2$

Use both sides as exponents on 10.

$10^{\log(x)} = 10^2$

Use inverse function properties.

$x = 10^2 = 100$