5.5

Properties of Logarithms

Apply basic properties of logarithms

Algebraic Properties of Logarithms

Common Logs		Natural Logs
$\log(1) = 0$		$\ln(1) = 0$
$\log(mn) = \log(m) + \log(n)$	⟷	$\ln(mn) = \ln(m) + \ln(n)$
$\log\left(\frac{m}{n}\right) = \log(m) - \log(n)$	⟷	$\ln\left(\frac{m}{n}\right) = \ln(m) - \ln(n)$
$\log(m^r) = r\log(m)$	⟷	$\ln(m^r) = r\ln(m)$

Log of a **product** is the sum of the individual logs.

Log of a **quotien**t is the log of the numerator minus the log of the denominator.

Log of an **expression raised to a power** is the power times the log of the expression.

IMPORTANT FACTS:

A logarithm is an exponent.

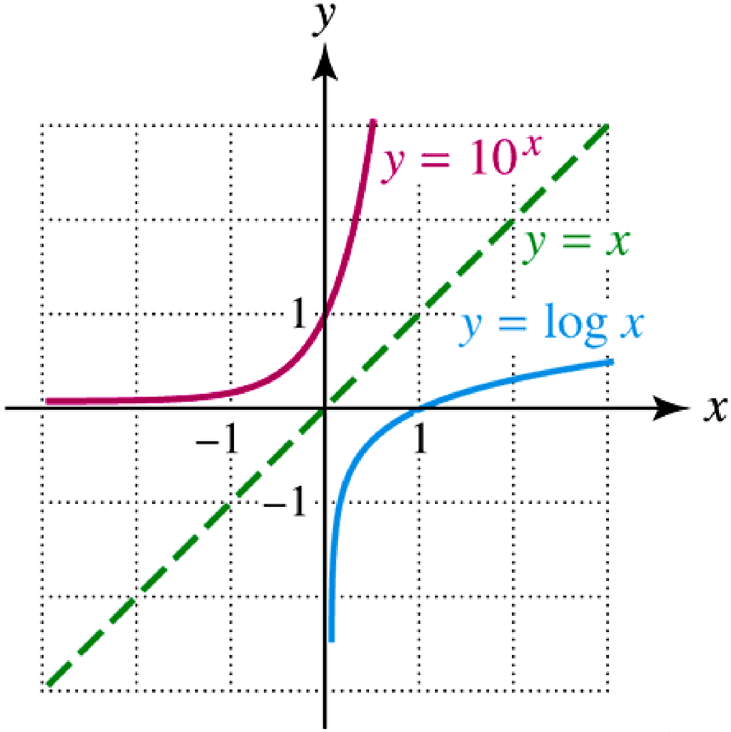
So $\log(10^w) = w$ and $\log(e^z) = z$ where w and z can be any real numbers.

Special cases:

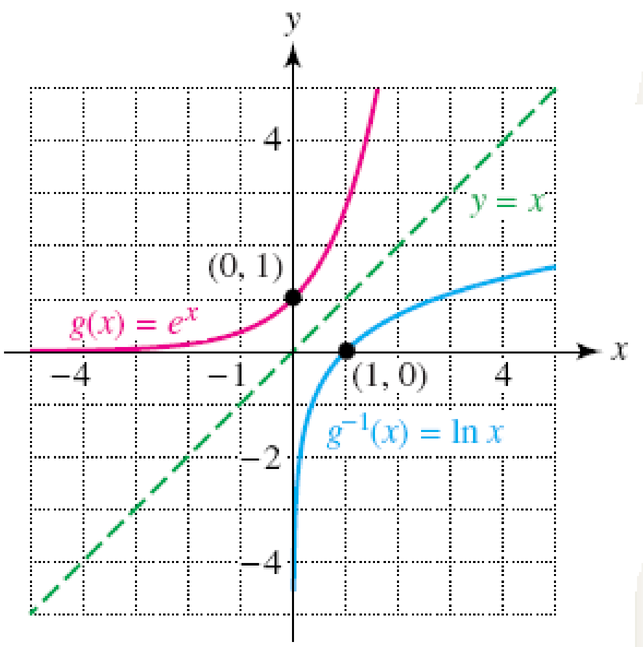
$$\log(10) = \log(10^1) = 1$$

$$\ln(e) = \ln(e^1) = 1$$

Let's see how these properties are developed. We look at the case for common logs.



Recall that functions log(x) and ln(x) have domain all positive real numbers.



Let m and n be positive real numbers, then since $y = 10^x$ has range all real numbers there exist exponent c and d so that $m = 10^c$ and $n = 10^d$.

So m and n be positive real numbers and we can express them as $\mathbf{m = 10^c}$ and $\mathbf{n = 10^d}$.

Let m = 1, then $m = 10^0$.

Then log(m) = log(1) = $\log(10^0)$. Since common logarithms are exponents on base 10, we have **log(1) = 0**.

Now consider the property **log(mn)=log(m) +log(n).**

So log(mn) = $\log(10^c \times 10^d) = \log(10^{c+d}) = c + d$ since a common logarithm is an exponent on base 10.

Next recall that log(m) = $\log(10^c)$ = c and log(n) = $\log(10^d)$ = d.

So we get **log(mn) = c + d = log(m) + log(n)**

The other properties are verified in a similar manner.

Examples:

$$\log(1) = 0 \qquad \ln(1) = 0$$

$$\log(mn) = \log(m) + \log(n) \qquad \ln(mn) = \ln(m) + \ln(n)$$

$$\log\left(\frac{m}{n}\right) = \log(m) - \log(n) \qquad \ln\left(\frac{m}{n}\right) = \ln(m) - \ln(n)$$

$$\log(m^r) = r\log(m) \qquad \ln(m^r) = r\ln(m)$$

(a) $\log(20) = \log(4\times5)$

$= \log(4) + \log(5)$

(b) $\log(600) = \log(6\times100)$

$= \log(6) + \log(100)$

$= \log(6) + \log(10^2) = \log(6) + 2$

(c) $\log(9/22) = \log(9) - \log(22)$

(d) $\log(6^{3.2}) = 3.2 \log(6)$

(a) $\ln(52) = \ln(4\times13) = \ln(4) + \ln(13)$

(b) $\ln(43/68) = \ln(43) - \ln(68)$

(c) $\ln(7.3^{-2.1}) = -2.1 \ln(7.3)$

Working with logarithms of expressions.

$$\log(1) = 0 \qquad \ln(1) = 0$$

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$$\log\left(\frac{m}{n}\right) = \log(m) - \log(n) \qquad \ln\left(\frac{m}{n}\right) = \ln(m) - \ln(n)$$

$$\log(m^r) = r\log(m) \qquad \ln(m^r) = r\ln(m)$$

Expand the expression and write it without exponents.

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(a) $\log(3x^5)$

(b) $\log\dfrac{\sqrt{x+9}}{(x-6)^4}$

(c) $\ln\left(\dfrac{4xe^x}{1-x}\right)$

Working with logarithms of expressions.

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$$\log(m^r) = r\log(m) \qquad \ln(m^r) = r\ln(m)$$

Expand the expression and write it without exponents.

(a) $\log(3x^5) = \log(3) + \log(x^5) = \log(3) + 5\log(x)$

(b) $\log\frac{\sqrt{x+9}}{(x-6)^4} = \log\sqrt{x+9} - \log(x-6)^4 = \frac{1}{2}\log(x+9) - 4\log(x-6)$

(c) $\ln\left(\frac{4xe^x}{1-x}\right) = \ln(4xe^x) - \ln(1-x) = \ln(4) + \ln(x) + \ln(e^x) - \ln(1-x)$

$= \ln(4) + \ln(x) + x - \ln(1-x)$

Application: Analyzing Sound with Decibels

Sound levels in decibels (db) can be computed by the expression $D(x) = 10 \log(10^{16}x)$.

(a) Use properties of logs to simplify the expression for D.

$D(x) = 10 \log(10^{16}x)$ ➔ $D(x) = 10 (\log(10^{16}) + \log(x))$

➔ $D(x) = 10 (16 + \log(x))$

➔ $D(x) = 160 + 10 \log(x)$

(b) Ordinary conversation has an intensity of $x = 10^{-10}$ w/cm2. Find the decibel level.

$D(10^{-10}) = 160 + 10 \log(10^{-10})$

$= 160 + 10(-10) = 160 -100 = 60$

Source	Intensity	Intensity Level
Threshold of Hearing (TOH)	$1*10^{-12}$ W/m^2	0 dB
Rustling Leaves	$1*10^{-11}$ W/m^2	10 dB
Whisper	$1*10^{-10}$ W/m^2	20 dB
Normal Conversation	$1*10^{-6}$ W/m^2	60 dB
Busy Street Traffic	$1*10^{-5}$ W/m^2	70 dB
Vacuum Cleaner	$1*10^{-4}$ W/m^2	80 dB
Large Orchestra	$6.3*10^{-3}$ W/m^2	98 dB
Walkman at Maximum Level	$1*10^{-2}$ W/m^2	100 dB
Front Rows of Rock Concert	$1*10^{-1}$ W/m^2	110 dB
Threshold of Pain	$1*10^{1}$ W/m^2	130 dB
Military Jet Takeoff	$1*10^{2}$ W/m^2	140 dB

http://www.glenbrook.k12.il.us/GBSSCI/PHYS/CLASS/sound/u11l2b.html

Using logarithms to simplify expressions.

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$$\log(m^r) = r\log(m) \qquad \ln(m^r) = r\ln(m)$$

(a) $\ln(2e) + \ln\left(\frac{1}{e}\right)$

(b) $\log(x^3) - \log(x^2)$

(c) $5\ln(x) + \ln(2x) - \ln(y)$

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(a) $\ln(2e) + \ln\left(\frac{1}{e}\right)$

$$\ln(2) + \ln(e) + \ln(e^{-1}) = \ln(2) + 1 + (-1)\ln(e) = \ln(2) + 1 - 1 = \ln(2)$$

(b) $\log(x^3) - \log(x^2)$ $\quad 3\log(x) - 2\log(x) = \log(x)$

(c) $6\ln(x) + \ln(2x) - \ln(y)$

$$\ln(x^6) + \ln(2x) - \ln(y) = \ln(x^6 \times 2x) - \ln(y) = \ln(2x^7) - \ln(y) = \ln\left(\frac{2x^7}{y}\right)$$

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Write as the logarithm of a single expression.

$$\log(21) + \log(8) - \log(6)$$

$$\log(21*8) - \log(6)$$

$$\log\left(\frac{21*8}{6}\right)$$

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Write as the logarithm of a single expression.

$$2\ln x - 4\ln y + \frac{1}{2}\ln z$$

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$$\ln\left(x^2\right) - \ln\left(y^4\right) + \ln\left(z^{\frac{1}{2}}\right)$$

$$\ln\left(\frac{x^2}{y^4}\right) + \ln\left(z^{\frac{1}{2}}\right)$$

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$$\ln\left(x^2\right) - \ln\left(y^4\right) + \ln\left(z^{\frac{1}{2}}\right)$$

$$\ln\left(\frac{x^2}{y^4}\right) + \ln\left(z^{\frac{1}{2}}\right)$$

$$\ln\left(\frac{x^2 \times z^{\frac{1}{2}}}{y^4}\right) = \ln\left(\frac{x^2 \times \sqrt{z}}{y^4}\right)$$