5.4

Logarithmic Functions and Models

- Evaluate the common logarithm function
- Solve basic exponential and logarithmic equations
- Focus on logarithms with base 10 and e
- Solve general exponential and logarithmic equations

A logarithm is an exponent.

Common Logarithm (base 10)

The common logarithm of a positive number x, denoted log x, is defined by

$$\log x = k \text{ if and only if } x = 10^k$$

where k is a real number.

(So the common logarithm of x is the exponent used on base 10 to produce the value x.)

The function given by $f(x) = \log x$ is called the common logarithm function.

Evaluate each of the following.

- log10
- log 100
- log 1000
- log 10000
- log (1/10)
- log (1/100)
- log (1/1000)
- log 1

= 1 because $10^1 = 10$

= 2 because $10^2 = 100$

= 3 because $10^3 = 1000$

= 4 because $10^4 = 10000$

= –1 because $10^{-1} = 1/10$

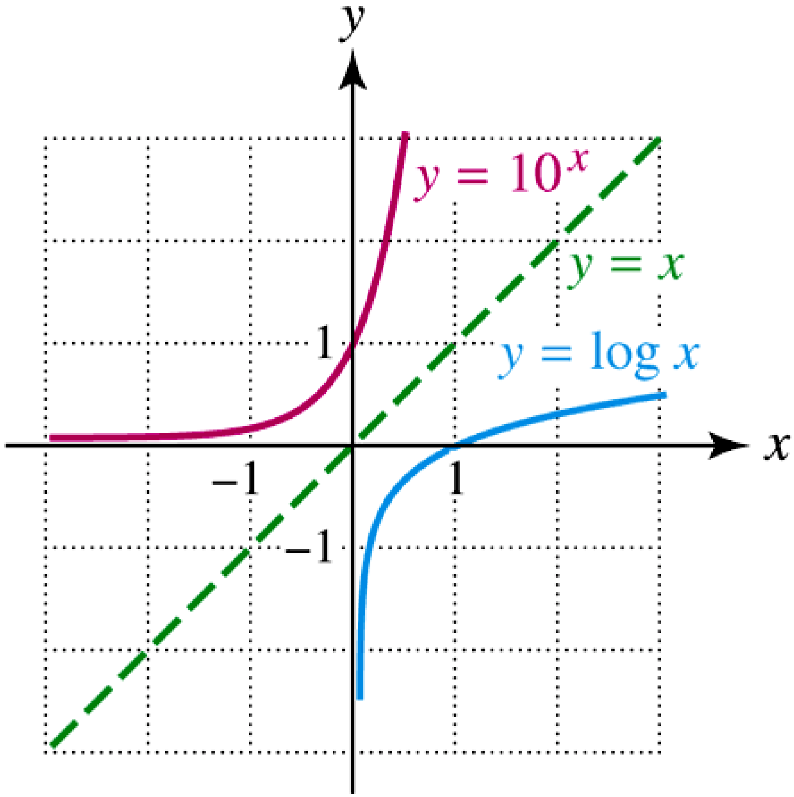
= –2 because $10^{-2} = 1/100$

= –3 because $10^{-3} = 1/1000$

= 0 because $10^0 = 1$

Graph of $f(x) = \log x$

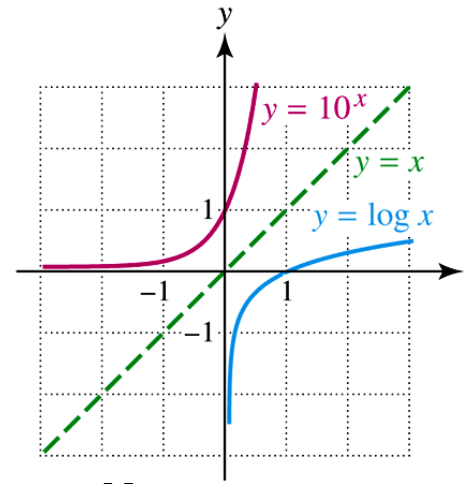
x	$log(x)$
.01	-2
.1	-1
1	0
10	1
100	2



Note that the graph of $y = \log x$ is the graph of $y = 10^x$ reflected through the line $y = x$. This suggests that these are inverse functions.

The domain of y = log(x) is all real numbers greater than zero; its range is all real numbers.

The Inverse of $y = \log x$



- Note that the graph of $f(x) = \log x$ passes the horizontal line test so it is a 1-1 function and has an inverse function.
- Find the inverse of $y = \log x$
- Using the definition of common logarithm to solve for x gives
- $x = 10^y$ ← It is the exponent to use on 10 to get x.
- Interchanging x and y gives
- $y = 10^x$
- So yes, the inverse of $y = \log x$ is $y = 10^x$

Inverse Properties of the Common Logarithm

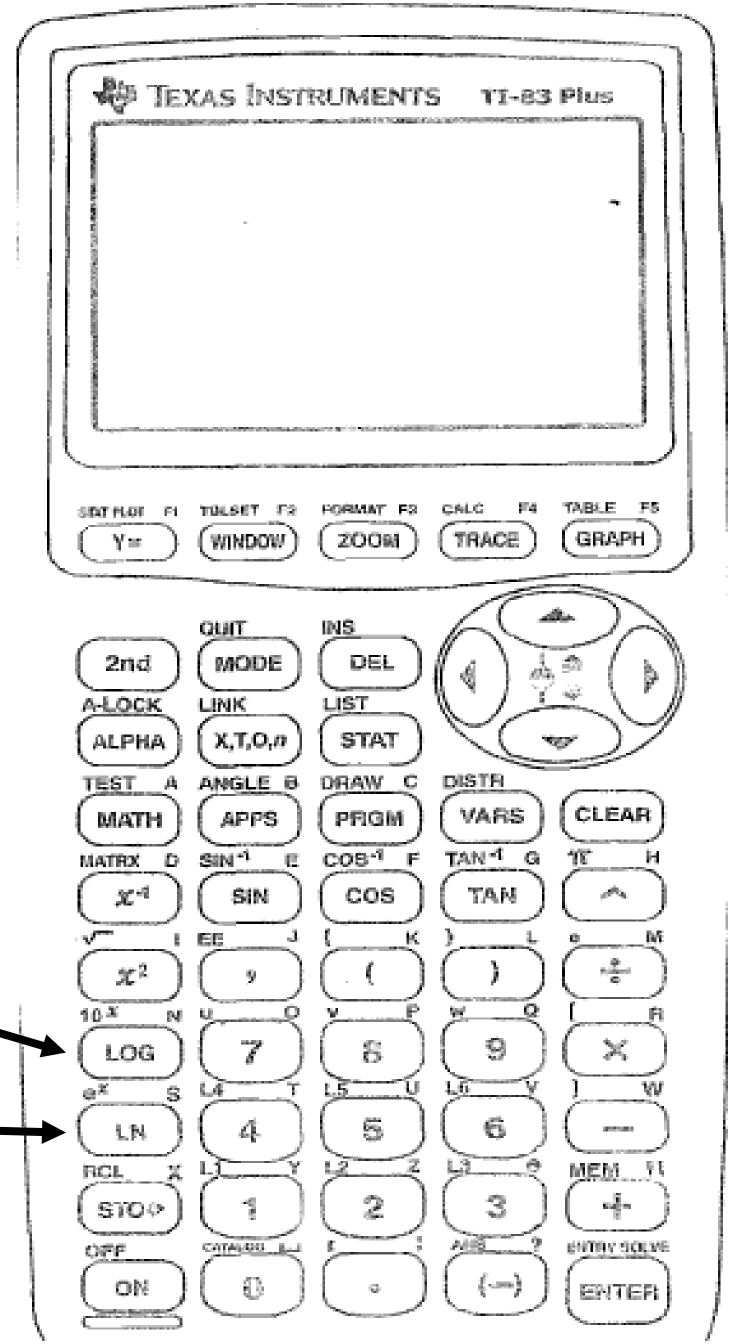
- For $f(x) = \log(x)$ we have that $f^{-1}(x) = 10^x$
- Since $(f \circ f^{-1})(x) = x$ for every x in the domain of f^{-1}
  - $\log(10^x) = x$ for all real numbers x.

- Since $(f^{-1} \circ f)(x) = x$ for every x in the domain of f
  - $10^{\log(x)} = x$ for any positive number x

Calculators and logarithms

Common logarithms

Natural logarithms



Solving Exponential Equations Using The Inverse Property $\log(10^x) = x$

- Solve the equation $10^x = 35$
- Take the common log of both sides
  - $\log(10^x) = \log(35)$
- Using the inverse property $\log(10^x) = x$ this simplifies to
  - $x = \log(35)$
- Using the calculator to estimate log (35) we have
  - $x \approx 1.54$

Solving Logarithmic Equations Using The Inverse Property $10^{\log x} = x$

- Solve the equation $\log(x) = 4.2$
- Use each side as an exponent on base 10
  - $10^{\log(x)} = 10^{4.2}$
- Using the inverse property $10^{\log(x)} = x$ this simplifies to
  - $x = 10^{4.2}$
- Using the calculator to estimate $10^{4.2}$ we have
  - $x \approx 15848.93$

Definition of Logarithm With Base *e*

Recall e ≈ 2.71828…

- **The logarithm with base *e* of a positive number *x*, denoted by ln(*x*) is defined by**

$$\ln(x) = k \text{ if and only if } x = e^k$$

where k is a real number.

(So the natural logarithm of x is the exponent used on base e to produce the value x.)

The function given by $f(x) = \ln(x)$ is called the natural logarithmic function.

Why is called the natural logarithm?

Why is it called the natural logarithm?

Intuitive explanation: **The natural log gives you the time needed to reach a certain level of growth**.

Suppose you have an investment with an interest rate of 100% per year, growing continuously. ($A = Pe^{rt}$) If you want 10 fold growth, **assuming continuous compounding**, you would have r = 1 and only need that t = ln(10) or 2.302 years so the result would be 10P.

e and the Natural Log are "twins":

e^x is the amount of continuous growth after a certain amount of time x.

Natural Log (ln) is the amount of **time** needed to reach a certain level of continuous growth.

http://betterexplained.com/articles/demystifying-the-natural-logarithm-ln/

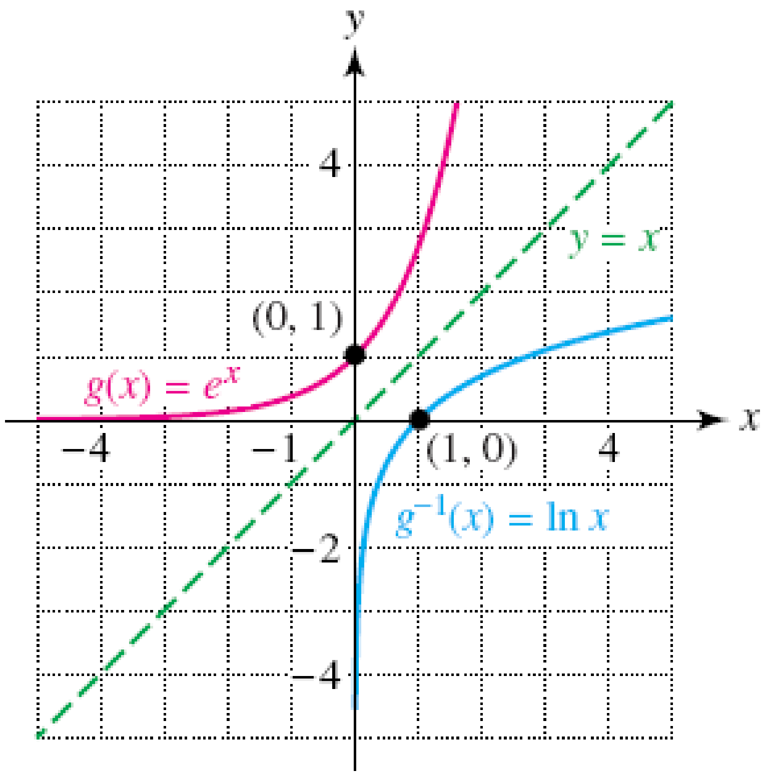
Evaluate each of the following without calculator.

- $\ln(e)$
- $\ln(e^2)$
- $\ln(1)$
- $\ln\sqrt{e}$

- $\ln(e) = 1$ since $e^1 = e$
- $\ln(e^2) = 2$ since 2 is the exponent that goes on e to produce e^2.
- $\ln(1) = 0$ since $e^0 = 1$
- $\ln\sqrt{e} = 1/2$ since 1/2 is the exponent used on e to produce $e^{1/2}$

Graph of $y = \ln(x)$

x	$ln(x)$
e^{-2}	-2
e^{-1}	-1
1	0
e	1
e^{2}	2



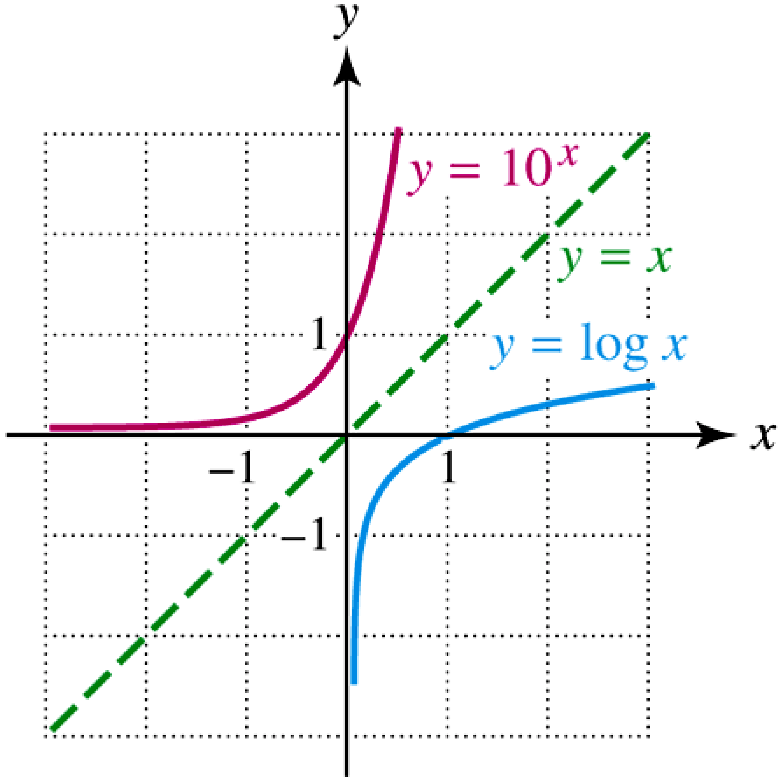
Note that the graph of $y = \ln(x)$ is the graph of $y = g(x) = e^x$ reflected through the line $y = x$. This suggests that these are inverse functions.

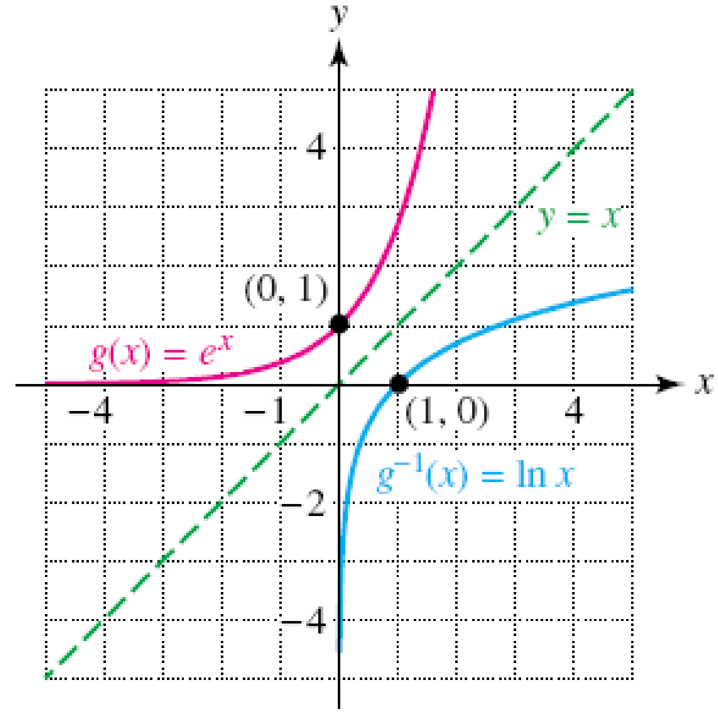
Inverse Properties of the Natural Logarithm

- **Recall that $f^{-1}(x) = e^x$ given $f(x) = \ln(x)$**
- **Since $(f \circ f^{-1})(x) = x$ for every x in the domain of f^{-1}**
  - **$\ln(e^x) = x$ for all real numbers x.**
- **Since $(f^{-1} \circ f)(x) = x$ for every x in the domain of f**
  - **$e^{\ln(x)} = x$ for any positive number x**

Solving Logarithmic Equations Using The Inverse Property $e^{\ln(x)} = x$

- **Solve the equation $\ln(x) = 1.5$**
- **Use both sides as an exponent on base *e***
  - $e^{\ln(x)} = e^{1.5}$
- **Using the inverse property $e^{\ln(x)} = x$ this simplifies to**
  - $x = e^{1.5}$
- **Using the calculator to estimate $e^{1.5}$**
  - $x \approx 4.48$





The domain of $y = 10^x$ and $y = e^x$ is all real numbers. The range of both is all real numbers greater than zero.

The domain of y = log(x) and y = ln(x) is all real numbers greater than zero. The range of both is all real numbers.

Recall properties of inverse functions.

Functions $y = 10^x$, $y = e^x$, $y = \log(x)$, and $y = \ln(x)$ can appear within composite functions as well as equations. Here are some examples.

Find the domain of each of the following:

(a) $y = e^{x^2-1}$ (b) $y = \log(3x+1)$ (c) $y = \ln(4x-3)$

Solve each of the following for x.

(a) $10^x - 5 = 120$ (b) $6 - \log(2x) = 3$

(c) $5e^{-x} + 2 = 20$ (d) $5\ln(3x) + 6 = 12$

Solutions:

Find the domain of each of the following:

(a) $y = e^{x^2-1}$ (b) $y = \log(3x+1)$ (c) $y = \ln(4x-3)$

All reals $x > -1/3$ $x > 3/4$

Solve each of the following for x.

(a) $10^x - 5 = 120$ (b) $6 - \log(2x) = 3$

(c) $5e^{-x} + 2 = 20$ (d) $5\ln(3x) + 6 = 12$

(a) $x = \log(125) \approx 2.096$ (b) $x = 500$

(c) $x = -\ln(18/5) \approx -1.281$ (d) $x = (1/3)e^{6/5} \approx 1.107$

Example:

Suppose that a person's salary is initially \$30,000 and is modeled by f(x) = 30,000 log(80 + x), where x represents the number of years of experience. If her current salary is \$60,000 determine the number of years of experience.

Solution: **Solve 60,000 = 30,000 log(80 + x) for x.**

$60,000 = 30,000\ \log(80 + x)$ divide both sides by 30,000

$2 = \log(80 + x)$ now compute $10^2 = 10^{\log(80+x)}$

We get $100 = 80 + x$, so $x = 20$

Example:

A pot of boiling water with temperature of 100°C is set in a room with a temperature of 20°C. The temperature T of the water after x hours is given by $T(x) = 20 + 80e^{-x}$.

(a) **Estimate the temperature of the water after 1 hour.**

(b) **How long did it take the water to cool to 60°C?**

Example:

A pot of boiling water with temperature of 100°C is set in a room with a temperature of 20°C. The temperature T of the water after x hours is given by $T(x) = 20 + 80e^{-x}$.

(a) Estimate the temperature of the water after 1 hour.

Solution: Compute T(1). We get about 49.4°C.

(b) How long did it take the water to cool to 60°C?

Solution: Solve $60 = 20 + 80e^{-x}$ for x.

$$40 = 80e^{-x}$$

$$1/2 = e^{-x} \Rightarrow \ln(1/2) = \ln(e^{-x}) = -x$$

$$\Rightarrow x = -\ln(1/2) \approx 0.69 \text{ hr} \approx 41.4 \text{ min}$$