5.2 Inverse Functions and Their Representations

- Calculate inverse operations
- Identify one-to-one functions
- Find inverse functions symbolically
- Use other representations to find inverse functions

Inverse Operations

Actions:

- Put on socks and put on shoes
- Put a gift inside a box and wrap the box
- Multiply *x* by 3 and add 2
- Take cube root of *x* and subtract 1

Inverse Actions

- Take off shoes and take off socks
- Unwrap the box and take the gift out of the box
- Subtract 2 from x and divide by 3
- Add 1 to *x* and cube the result

Reminder of the definition of a function

- **y = $f(x)$ means that given an input x, there is just one corresponding output y.**

- **Graphically, this means that the graph passes the vertical line test.**

- **Numerically, this means that in a table of values for $y = f(x)$ there are no x-values repeated that have different y-values.**

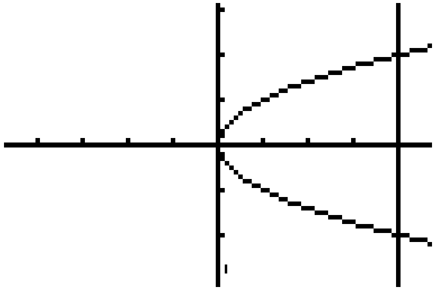
Example

- **Given $y^2 = x$, is $y = f(x)$?**

  That is, is y a function of x?

  - **No, because if $x = 4$, y could be 2 or –2.**
    - **Note that the graph fails the vertical line test.**

x	y
4	– 2
1	– 1
0	0
1	1
4	2



    - **Note that there is a value of x in the table for which there are two different values of y (that is, x-values are repeated.)**

Idea Behind a One-to-One Function

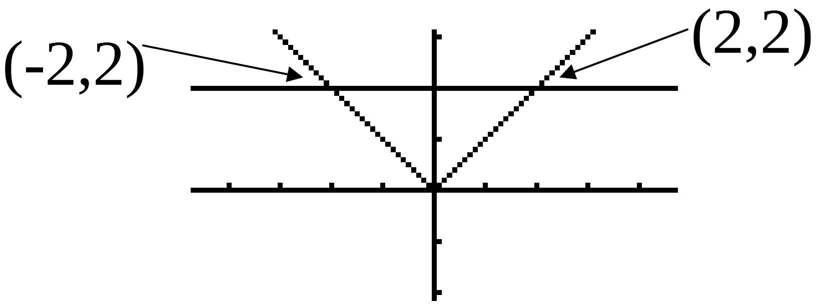
- Given a function $y = f(x)$, f is 1-1 (pronounced "one-to-one") means that given an output y there was just one input x which produced that output.
  - Graphically, this means that the graph passes the horizontal line test. (Every horizontal line intersects the graph at most once.)
  - Numerically, this means the there are no y-values repeated in a table of values.

Example

Given $y = f(x) = |x|$, is f 1-1?

No, because if $y = 2$, x could be 2 or – 2.

- **Note that the graph fails the horizontal line test.**



Examples

Is $y = x^3 - 4$ one-to-one?

Is $y = x^4$ one-to-one?

Formal Definition of One-to-One Function

A function f is a **one-to-one function** if, for elements c and d in the domain of f,

$$c \neq d \text{ implies } f(c) \neq f(d)$$

Given a 1-1 function f

f^{-1} is a symbol for the inverse of the function f, not to be confused with the reciprocal.

If $f^{-1}(x)$ does <u>NOT</u> mean $1/f(x)$, what does it mean?

$$y = f^{-1}(x) \text{ means that } x = f(y)$$

Note that $y = f^{-1}(x)$ is pronounced "y equals f inverse of x."

$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$

Composition of functions.

This says f and f^{-1} "undo" one another.

FINDING a SYMBOLIC REPRESENTATION for an INVERSE FUNCTION

To find a formula for f^{-1}, perform the following steps.

Step 1. **Verify that f is a one-to-one function.**

Step 2. **Solve equation $y = f(x)$ for x; the formula you get will be x = some expression in y.**

Step 3. **Interchange x and y in the result from Step 2 to get expression $y = f^{-1}(x)$.**

Example:

Let F be Fahrenheit temperature and let C be Centigrade temperature then $F = f(C) = (9/5)C + 32$ converts Centigrade to Fahrenheit.

Find f^{-1}.

Step 1. Is f one-to-one?

Step 2. Let $y = f(x) = (9/5)x + 32$. Solve for x.

Step 3. Interchange the "names" x and y to get $y = f^{-1}(x)$.

Express the meaning of f^{-1} in words.

Compute $f^{-1}(100)$.

Example

Let $f(x)$ compute the distance traveled in miles after x hours by a car with a velocity of 60 miles per hour. Explain what f^{-1} computes.

We are given that distance = f(time) so time = f^{-1}(distance). f^{-1} computes the time it takes a car with a velocity of 60 mph to travel x miles.

Example

- **Describe verbally the inverse of the statement. Then express both the statement and its inverse symbolically.**
  - **Take the cube root of *x* and add 1.**
- **To undo taking a cube root and adding 1, we must subtract 1 and cube the result.**

Expressing the original statement symbolically

$$y = f(x) = \sqrt[3]{x} + 1$$

Expressing the inverse symbolically

$$y = f^{-1}(x) = (x - 1)^3$$

Formal Definition of an Inverse Function

Let f be a 1-1 function. Then f^{-1} is the inverse function of f, if

- $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ for every x in the domain of f
- $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}

Using composition of functions verify that if

$f(x)=\sqrt[3]{x}+1$ then $f^{-1}(x)=(x-1)^3$

<u>Step 1</u>

$$(f^{-1}\circ f)(x)=f^{-1}(f(x))=$$

$$f^{-1}(\sqrt[3]{x}+1)=((\sqrt[3]{x}+1)-1)^3=(\sqrt[3]{x})^3=x$$

Using composition of functions verify that if

$$f(x) = \sqrt[3]{x} + 1 \quad \text{then} \quad f^{-1}(x) = (x-1)^3$$

Step 2

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) =$$

$$f((x-1)^3) = \sqrt[3]{(x-1)^3} + 1 = (x-1) + 1 = x$$

Why does *f* need to be 1-1 to have an inverse function?

Suppose $f(x) = x^2$

- **$f(3) = 9$ and $f(-3) = 9$**
- **If we defined $f^{-1}(9)$, it could have two different values, namely 3 and –3 and thus f^{-1} would not be a function.**

Evaluating an inverse function numerically

x	$f(x)$
1	–5
2	–3
3	0
4	3
5	5

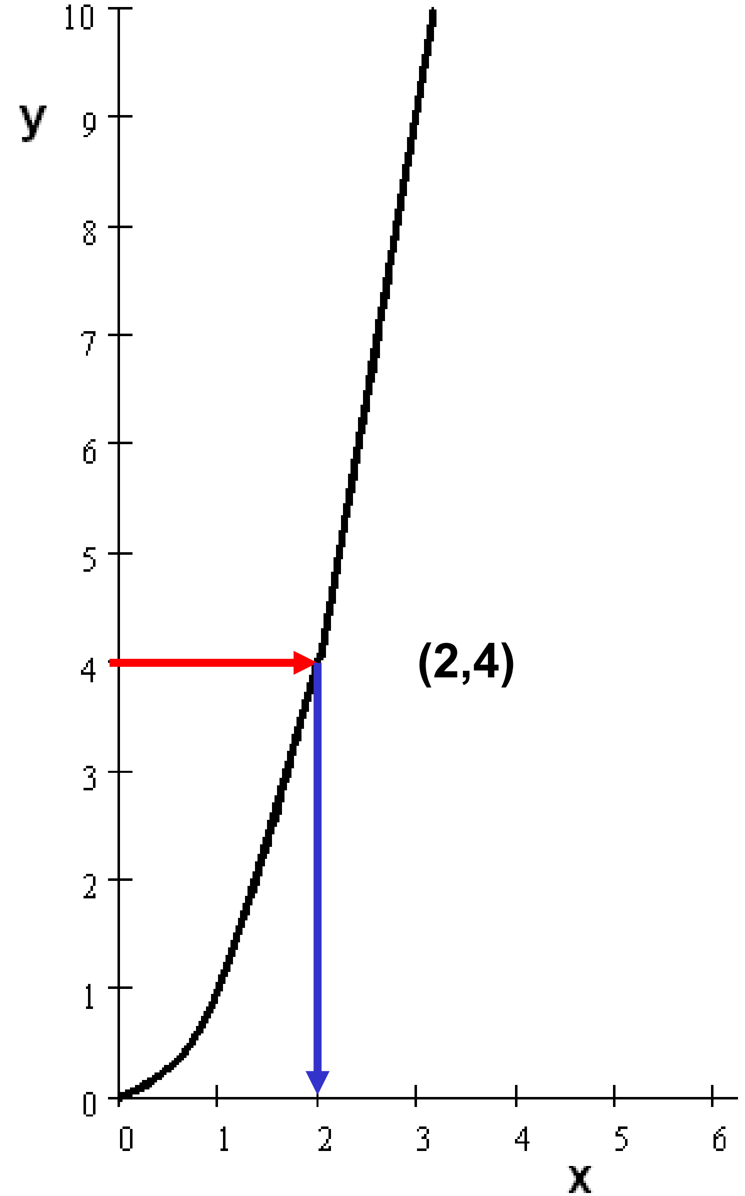
- **The function is 1-1 so f^{-1} exists.**
- $f^{-1}(-5) = 1$
- $f^{-1}(-3) = 2$
- $f^{-1}(0) = 3$
- $f^{-1}(3) = 4$
- $f^{-1}(5) = 5$

Evaluating an inverse function graphically

- The graph of *f* passes the horizontal line test so *f* is 1-1.
  Evaluate $f^{-1}(4)$.

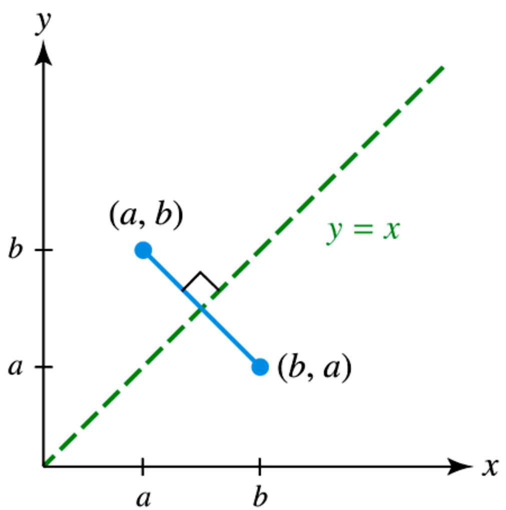
  So $f^{-1}(4) = 2$

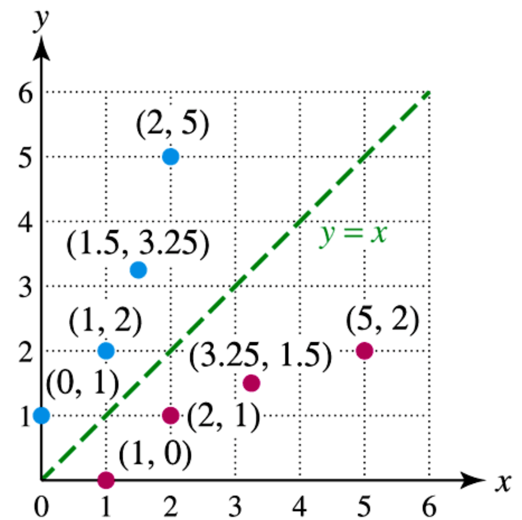
- Since the point (2,4) is on the graph of *f*, the point (4,2) will be on the graph of f^{-1} since $f^{-1}(4) = 2$.

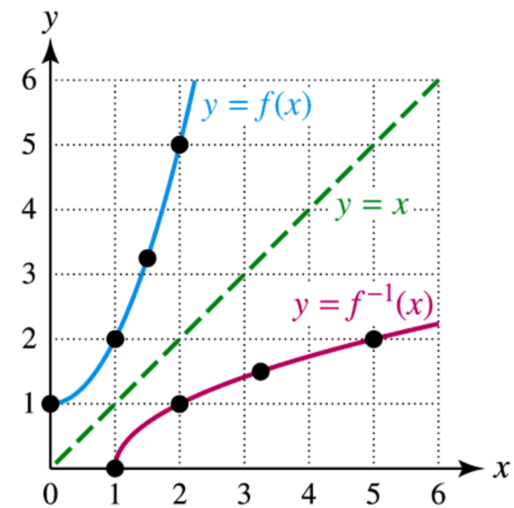


Graphs of Functions and Their Inverses

- **The graph of f^{-1} is a reflection of the graph of f across the line $y = x$**







Note that the domain of f equals the range of f^{-1} and the range of f equals the domain of f^{-1} . (We interchange domain & range.)