4.7

Power Functions and Radical Equations

- Review properties of rational exponents
- Learn radical notation
- Understand properties and graphs of power functions
- Use power functions to model data
- Solve equations involving rational exponents
- Solve equations involving radical expressions

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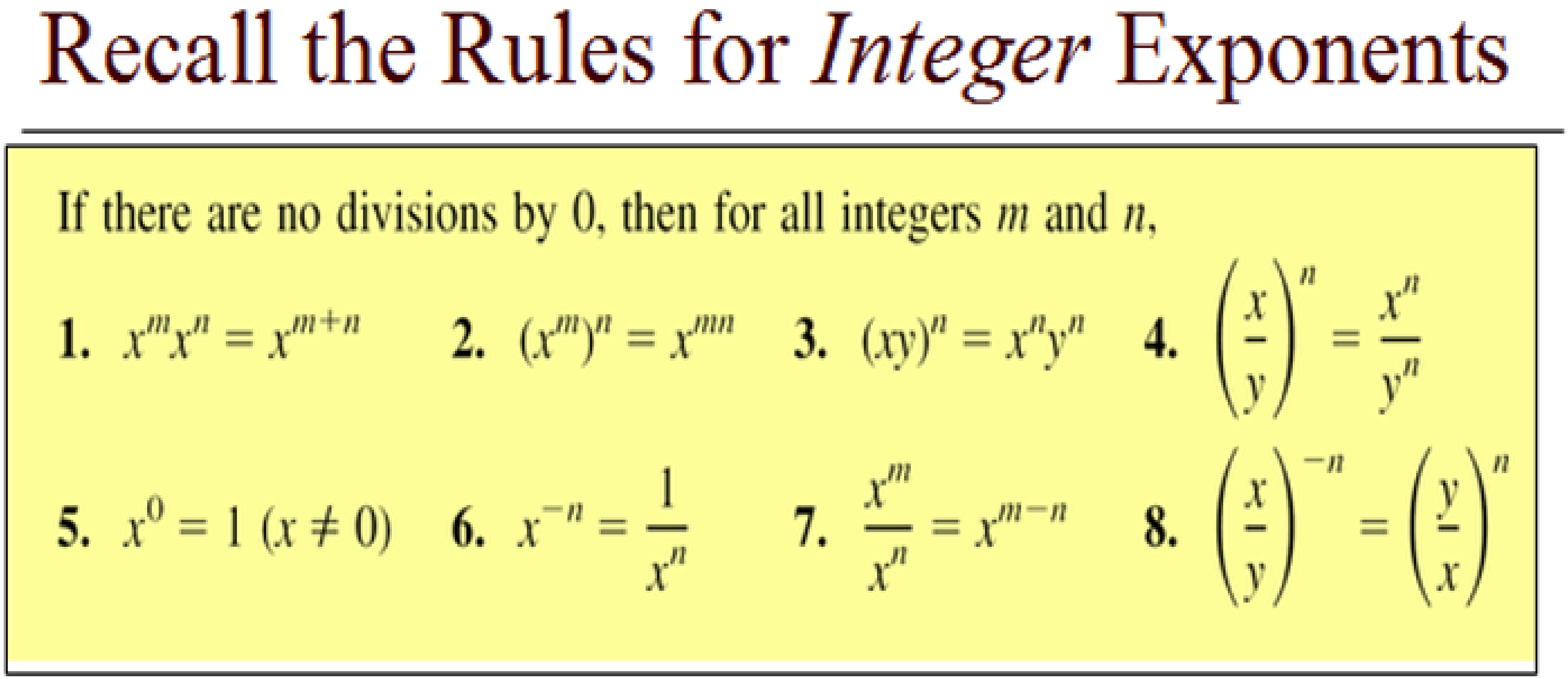
Addison Wesley

The same rules apply for rational exponents.



Recall the Rules for *Integer* Exponents

If there are no divisions by 0, then for all integers m and n,

1. $x^m x^n = x^{m+n}$

2. $(x^m)^n = x^{mn}$

3. $(xy)^n = x^n y^n$

4. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

5. $x^0 = 1 \ (x \neq 0)$

6. $x^{-n} = \frac{1}{x^n}$

7. $\frac{x^m}{x^n} = x^{m-n}$

8. $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$

Property

1. $b^{m/n} = (b^m)^{1/n} = (b^{1/n})^m$

2. $b^{m/n} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$

3. $(b^r)^p = b^{rp}$

4. $b^{-r} = \dfrac{1}{b^r}$

5. $b^r b^p = b^{r+p}$

6. $\dfrac{b^r}{b^p} = b^{r-p}$

Computational Examples

Simplify each expression by hand.

a) $8^{2/3}$

b) $(-32)^{-4/5}$

Solutions

$$a)\ 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$$

$$b)\ (-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

More Examples

Use positive rational exponents to write each expression.

a) $\sqrt[5]{x^4}$

b) $\sqrt{\sqrt[3]{x} \bullet \sqrt[6]{x}}$

Solutions:

$$a)\ \sqrt[5]{x^4} = \left(x^4\right)^{1/5} = x^{4/5}$$

$$b)\ \sqrt{\sqrt[3]{x} \bullet \sqrt[6]{x}} = \left(x^{1/3} \bullet x^{1/6}\right)^{1/2} = \left(x^{(1/3)+(1/6)}\right)^{1/2}$$

$$= \left(x^{1/2}\right)^{1/2} = x^{1/4}$$

Power function models

A power function model will have an equation like $y = ax^b$, where a is real coefficient and b will be a rational number $\frac{m}{n}$ in lowest terms with $n \geq 2$.

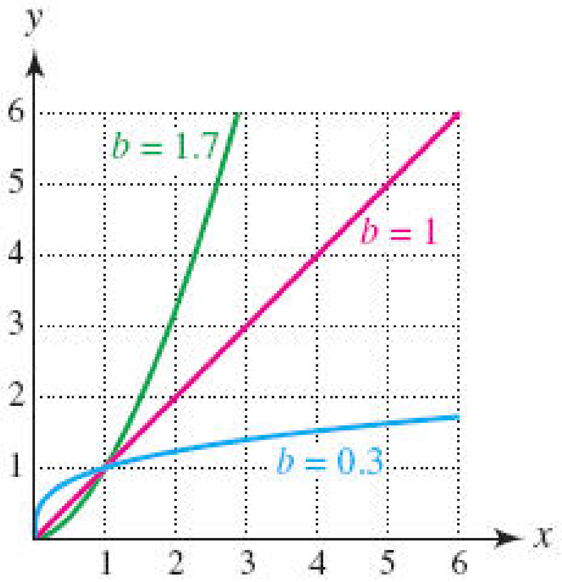
If $b = \frac{1}{n}$ for some integer $n \geq 2$, then x^b is a root function.

Examples:

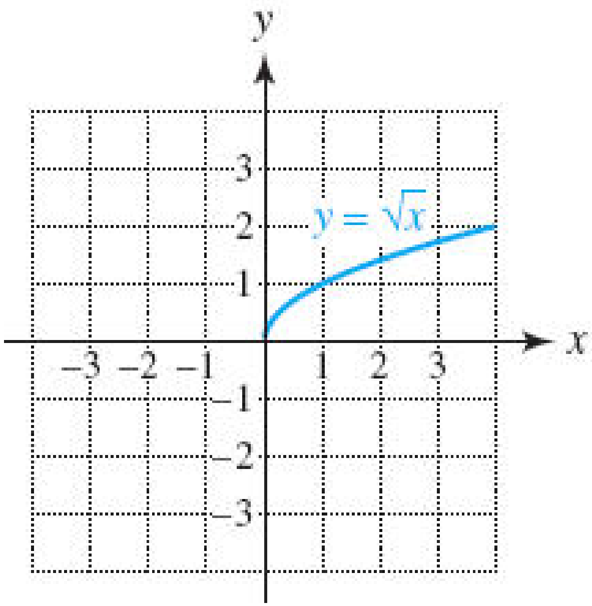
$y = a\sqrt{x}$, $y = ax^2$, $y = ax^{3/4}$, $y = a\sqrt[3]{x}$, $y = ax^{0.27}$, $y = a\sqrt[4]{x^5}$

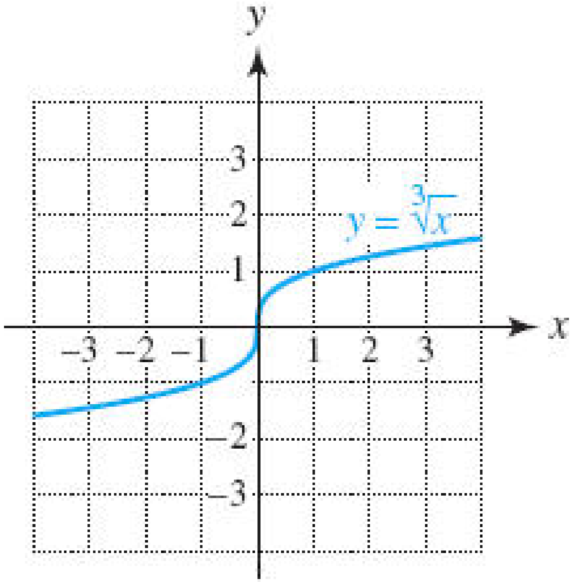
The domain of $x^{p/q}$ is all reals when q is odd and all nonnegative reals when q is even.

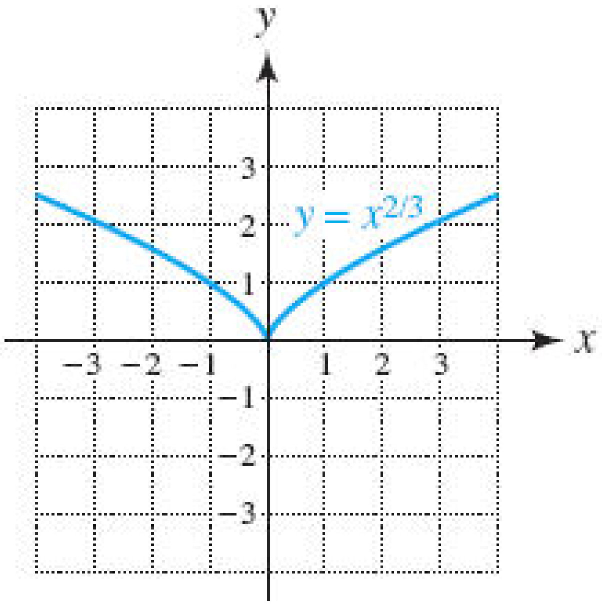
Graphs of some power functions



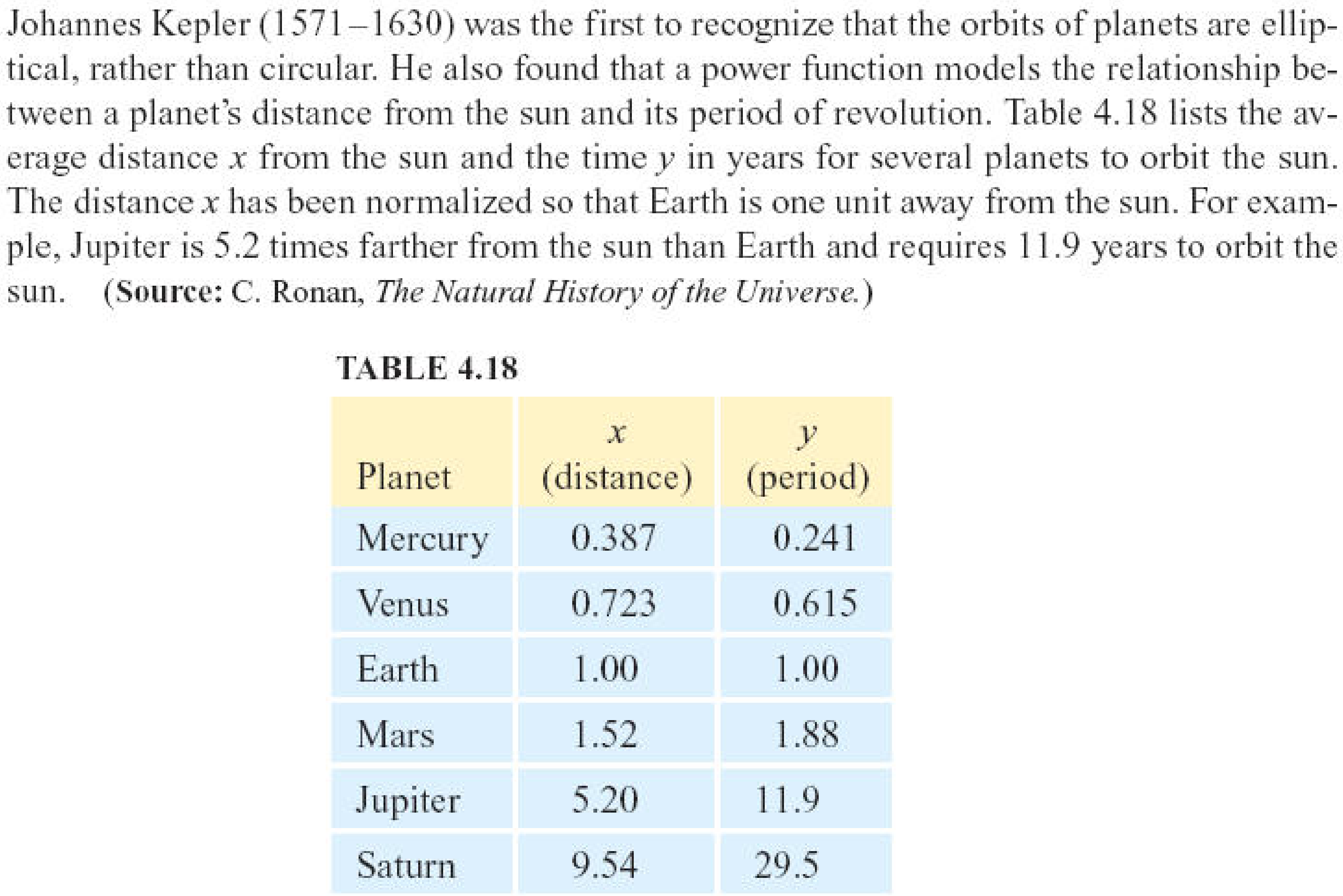
$f(x) = x^b, x \geq 0$







Planets, orbits, and distance from the sun.

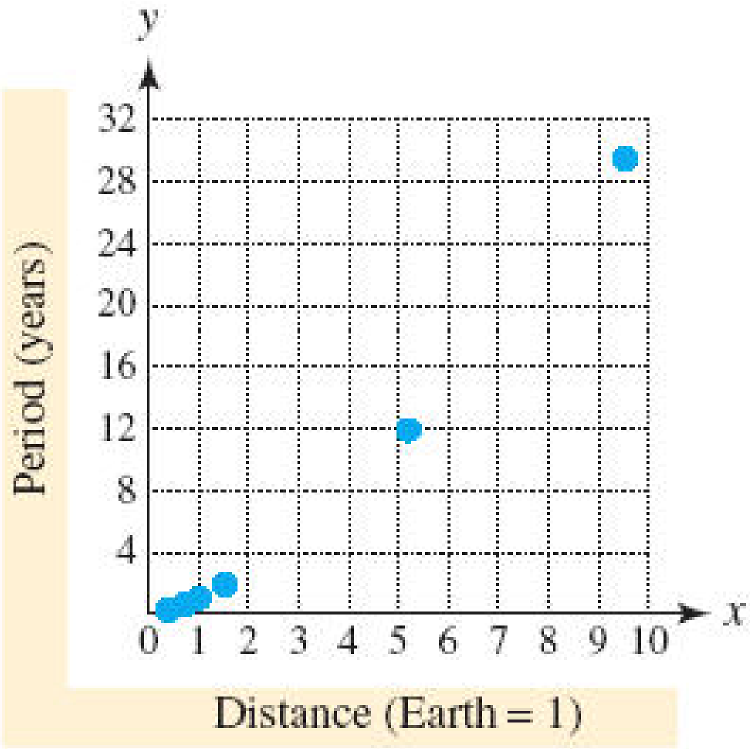


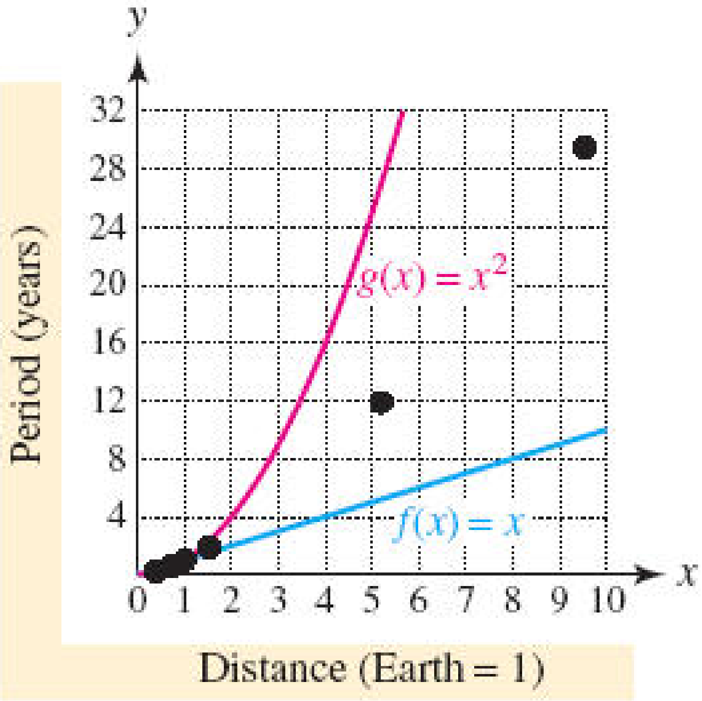
Johannes Kepler (1571–1630) was the first to recognize that the orbits of planets are elliptical, rather than circular. He also found that a power function models the relationship between a planet's distance from the sun and its period of revolution. Table 4.18 lists the average distance x from the sun and the time y in years for several planets to orbit the sun. The distance x has been normalized so that Earth is one unit away from the sun. For example, Jupiter is 5.2 times farther from the sun than Earth and requires 11.9 years to orbit the sun. (**Source:** C. Ronan, *The Natural History of the Universe.*)

TABLE 4.18

Planet	x (distance)	y (period)
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.00	1.00
Mars	1.52	1.88
Jupiter	5.20	11.9
Saturn	9.54	29.5

Let's look at a scatter plot for the data.





It appears that a <u>polynomial</u> may be a good model.

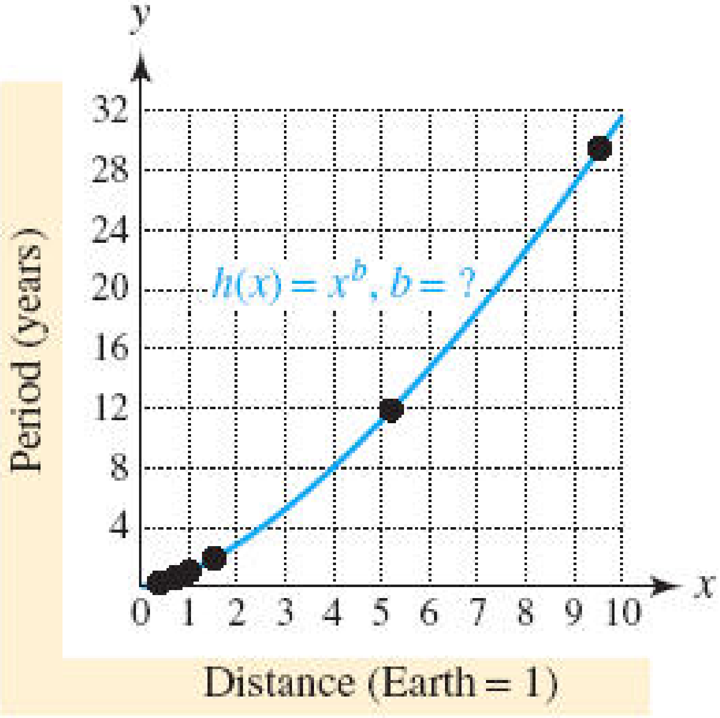
Trying $y = x$ and $y = x^2$.

Possibly try $y = x^b$, $1 < b < 2$.

$y = x^b$, b <u>any</u> real number is called a **power function**.

Consider a power function model to the data.

Approximate with a smooth curve <u>close</u> to the points.



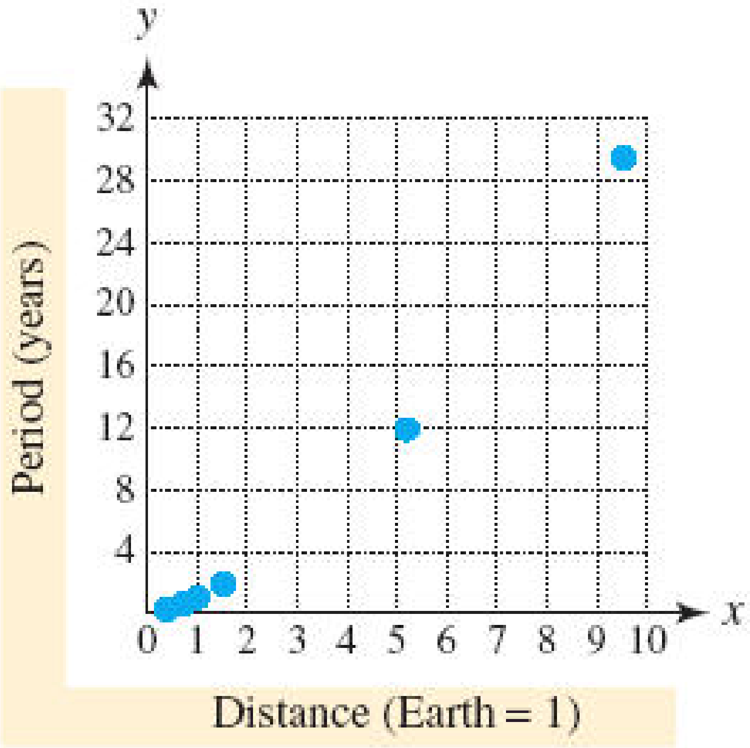
How do we determine an appropriate value for the exponent b?

<u>Two approaches:</u>

1. **Experiment with different choices for b, then check graphically and by constructing a table of values of the model.**
2. **Develop a regression scheme using power functions.**

Modeling the period of planetary orbits

Scatter plot



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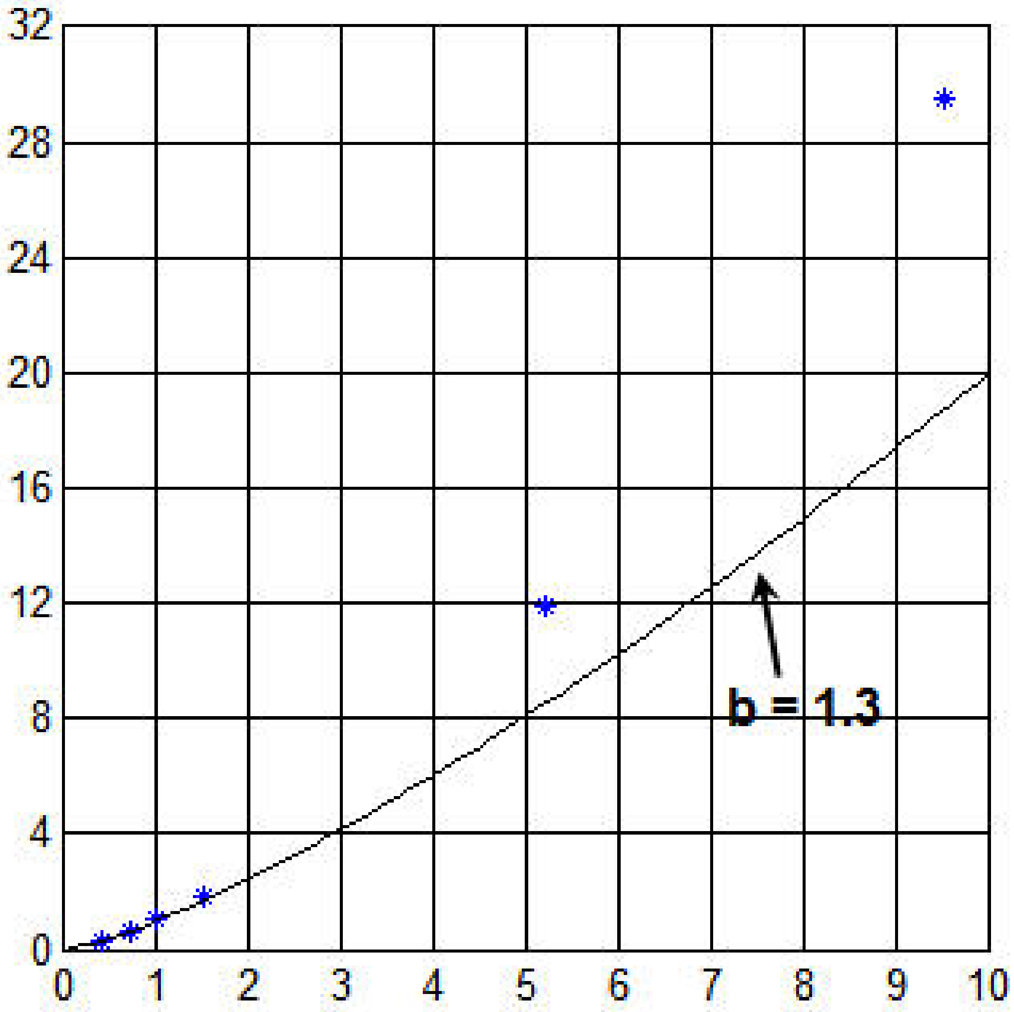
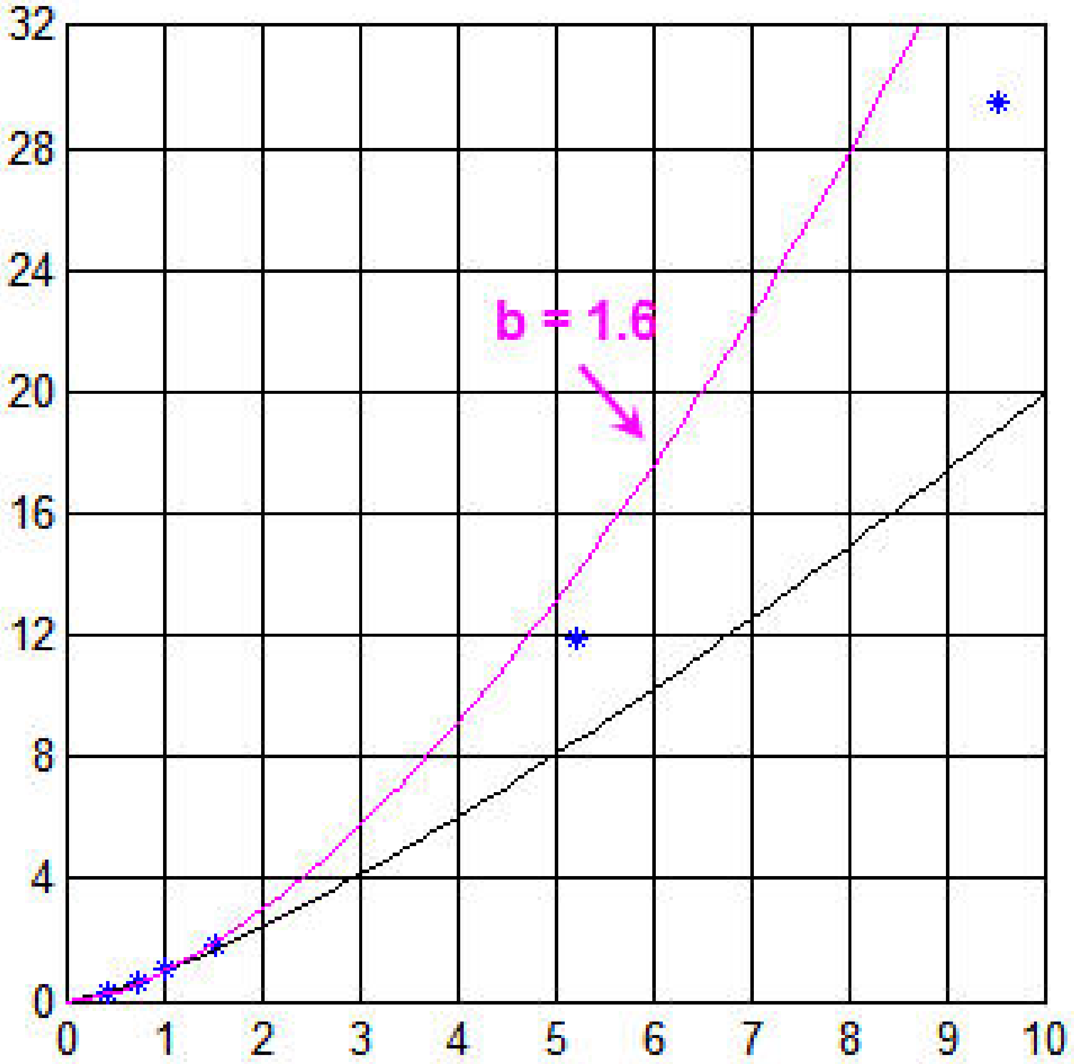
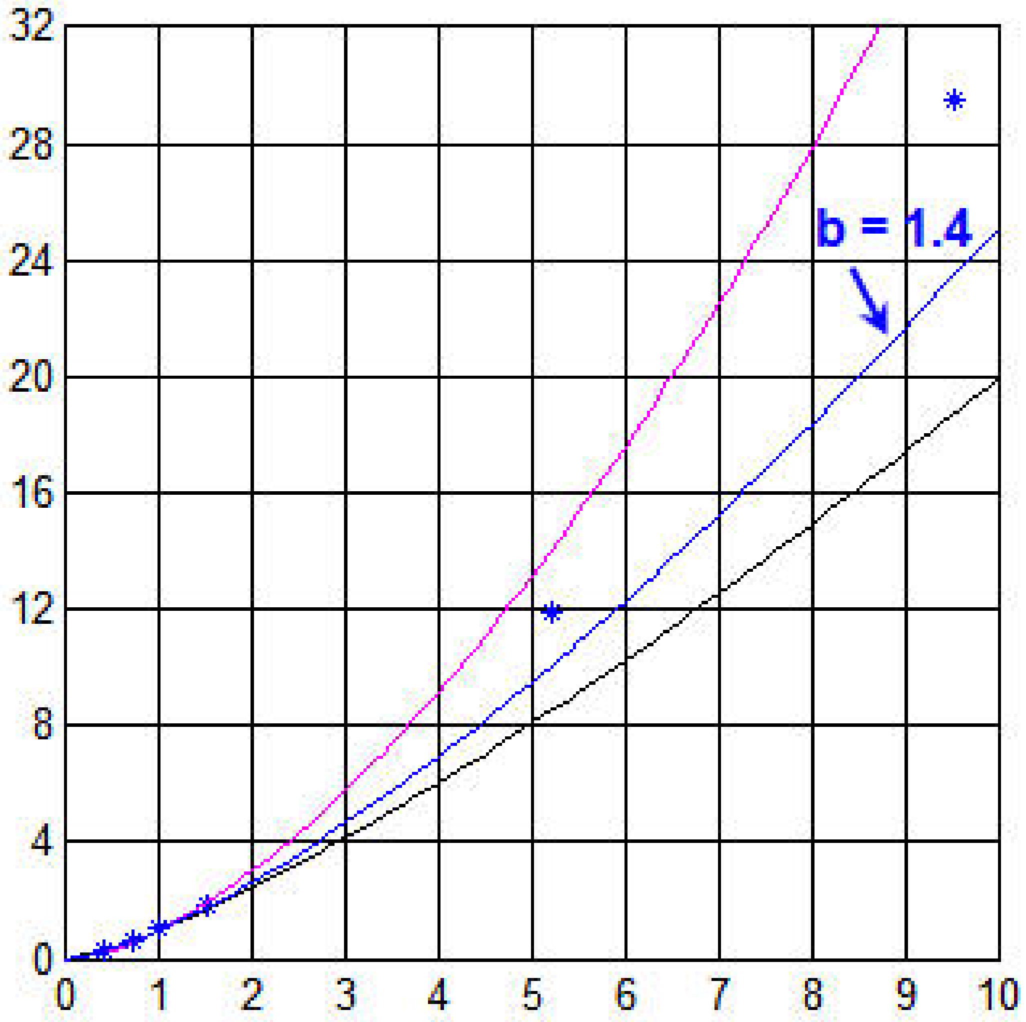
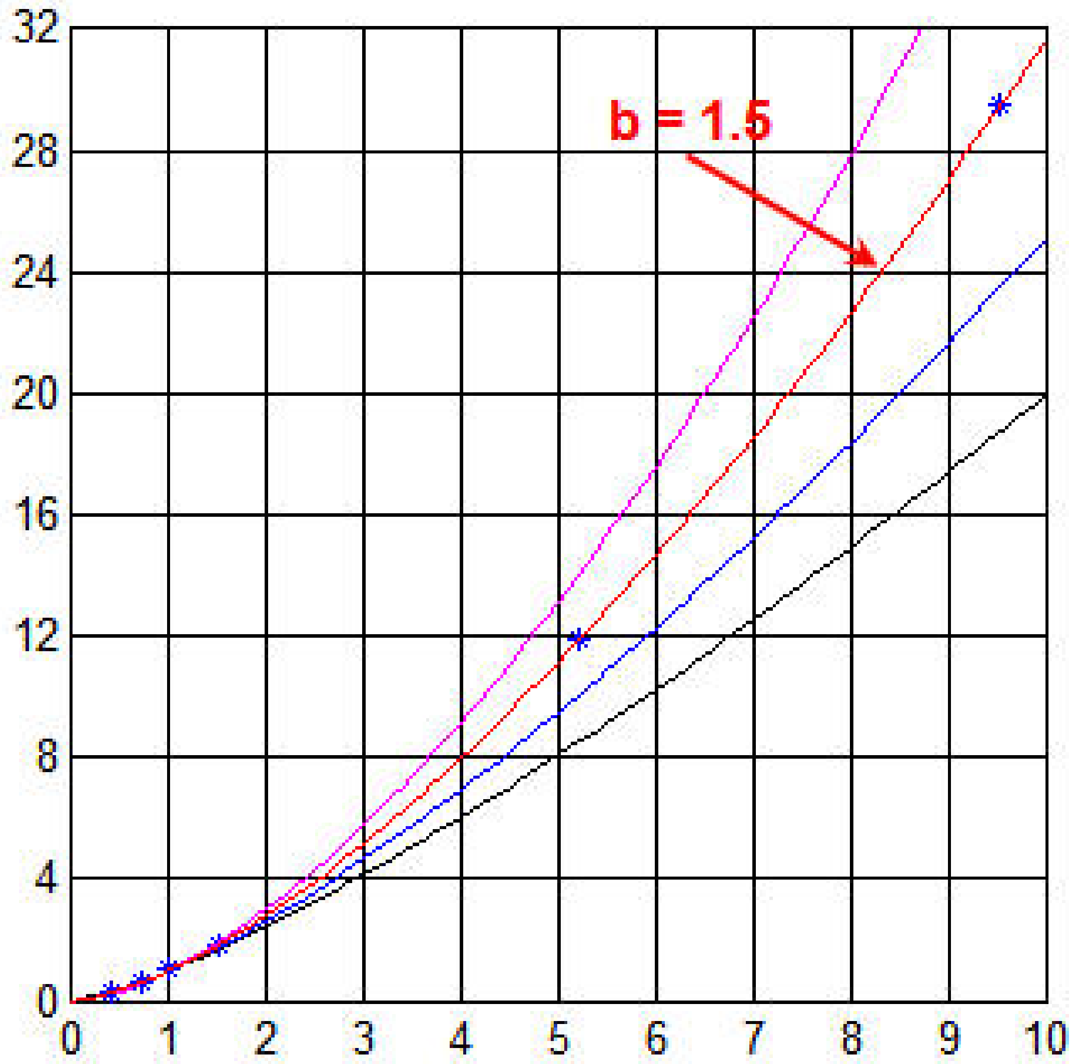
Experiment with exponents b in **$y = x^b$, $1 < b < 2$.**


32
28
24
20
16
12
8
4
0
0
1
2
3
4
5
6
7
8
9
10
b = 1.3


b = 1.6
32
28
24
20
16
12
8
4
0
0
1
2
3
4
5
6
7
8
9
10


32
28
24
20
16
12
8
4
0
0
1
2
3
4
5
6
7
8
9
10
b = 1.4


b = 1.5
0
4
8
12
16
20
24
28
32
0
1
2
3
4
5
6
7
8
9
10

Check the model $y = x^{1.5}$ against the original data.

Planet	X (distance)	Y (period)	Model prediction	Percent of Error
Mercury	0.387	0.241	0.2408	-.083%
Venus	0.723	0.615	0.6148	-.033%
Earth	1.00	1.00	1.0000	0
Mars	1.52	1.88	1.8740	-.32%
Jupiter	5.20	11.9	11.8578	-.035%
Saturn	9.54	29.5	29.4661	-.012%

So the model $y = x^{1.5}$ behaved quite well.

Let's do another type of check.

There is distance vs. period data available for other planets in our solar system.

Planet	x (distance)	y (period)	Model prediction	Percent of Error
Uranus	19.2	84.0	84.13	.015%
Neptune	30.1	164.8	165.14	.02%
Pluto	39.5	248.5	248.25	-.1%

Use the model $y = x^{1.5}$ to estimate the period and compute the relative error.

About the same type of accuracy as the model approximations to the original data.

Modeling Wing Size of a Bird

Heavier birds have larger wings with more surface areas than do lighter birds. For some species the relationship can be modeled by $S(w) = 0.2w^{2/3}$, where w is the weight of the bird in kilograms and S is surface area of the wings in square meters.

a) **Approximate $S(0.75)$ and interpret the result.**

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$S(0.75) = 0.2(0.75)^{2/3} \approx 0.165$ **square meter**

The wings of a bird that weighs about 0.75 kilogram have the surface area of about 0.165 square meter.

b) What weight corresponds to a surface area of 0.45 square meter?

To answer this, we must solve the equation $0.2w^{2/3} = 0.45$.

$$0.2w^{2/3} = 0.45 \quad \Rightarrow \quad w^{2/3} = \frac{0.45}{0.2} \quad \Rightarrow \quad \left(w^{2/3}\right)^3 = \left(\frac{0.45}{0.2}\right)^3$$

$$\Rightarrow \quad w^2 = \left(\frac{0.45}{0.2}\right)^3 \quad \Rightarrow \quad w = \pm\sqrt{\left(\frac{0.45}{0.2}\right)^3} \quad \Rightarrow \quad w \approx \pm 3.375$$

Since w must be positive we have that the wings of a 3.375 kg bird have a surface area of about 0.25 sq. meters.

Equations Involving Rational Exponents

Example: Solve $4x^{3/2} - 6 = 6$. Approximate the answer to the nearest hundredth

$$4x^{3/2} - 6 = 6$$

$$4x^{3/2} = 12$$

$$(x^{3/2})^2 = 3^2$$

$$x^3 = 9$$

$$x = 9^{1/3}$$

$$x = 2.08$$

Equations Having Negative Exponents

Example: Solve $6x^{-2} + x^{-1} = 2.$

Let u = x^{-1}. Then we have

$$6x^{-2} + x^{-1} = 2$$

$$6u^2 + u - 2 = 0$$

BB

$$(3u + 2)(2u - 1) = 0$$

$$u = -\frac{2}{3} \quad \text{or} \quad u = \frac{1}{2}$$

$$x = -\frac{3}{2} \quad \text{or} \quad x = 2$$

Since $u = \frac{1}{x}$, then $x = \frac{1}{u}$.

Equations Involving Radicals

When solving equations that contain square roots, it is common to square each side of an equation.

Example: Solve $\sqrt{3x-2} = x-2.$

BB

$$\sqrt{3x-2} = x-2$$

$$\left(\sqrt{3x-2}\right)^2 = \left(x-2\right)^2$$

$$3x-2 = x^2-4x+4$$

$$x^2-7x+6=0$$

$$\left(x-1\right)\left(x-6\right)=0$$

$$x=1 \text{ or } x=6$$

WARNING: squaring both sides can introduce "extraneous solutions", so we need to check the value in the original problem.

Check in $\sqrt{3x-2}=x-2.$

$$\sqrt{3(1)-2}=1-2 \qquad \sqrt{3(6)-2}=6-2$$

$$1\neq -1 \qquad 4=4$$

Substituting these values in the original equation shows that the value of 1 is an extraneous solution because it does not satisfy the given equation.

Therefore, the only solution is 6.

Example:

Some equations may contain a cube root.

BB

Solve & $\sqrt[3]{4x^2-4x+1}=\sqrt[3]{x}.$

Check.

$$\sqrt[3]{4x^2-4x+1}=\sqrt[3]{x}$$

$$\left(\sqrt[3]{4x^2-4x+1}\right)^3=\left(\sqrt[3]{x}\right)^3$$

$$4x^2-5x+1=0$$

$$(4x-1)(x-1)=0$$

$$x=\frac{1}{4}\quad\text{or}\quad x=1$$

Both solutions check, so the solution set is $\left\{\frac{1}{4},\ 1\right\}.$