

4.3-4.4

Finding Real Zeros of Polynomial Functions

- ◆ Analyze polynomials with multiple zeros
- ◆ Approximation Methods
- ◆ Solve polynomial equations

To Solve polynomial equations we find solutions to

$$p(x) = a^n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

Degree 1: $ax + b = 0$, $a \neq 0$ is easy and there is one solution.

Degree 2: $ax^2 + bx + c = 0$, $a \neq 0$ can always be solved using the quadratic formula or we can try to factor it.

Warning: In some cases solutions may be complex numbers.

Examples:

$$x^2 - 1 = (x - 1)(x + 1)$$

$$x^2 + 1 = (x - i)(x + i), \text{ where } i = \sqrt{-1}$$

General case: $ax^2 + bx + c = 0$, $a \neq 0$ will have complex solutions if the discriminant $b^2 - 4ac < 0$

Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Higher degree polynomials: There are (complicated) formulas for degree three and degree four polynomials, but not for higher degrees. Factoring can be “tried”, but it is usually very difficult since some solutions may be complex. Often the solutions can be approximated to a good degree of accuracy.

Lots of luck needed!



Example: $p(x) = x^3 + 3x^2 + 4x + 12$

This polynomial factors as $(x + 3)(x^2 + 4)$

Solutions to polynomial equations are called “**zeros**” or “**roots**” of the polynomial. (These are the x-intercepts.)

$$\text{Let } p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0.$$

If $x = k$ is a zero of the polynomial, then $p(k) = 0$.

Linear equations $ax + b = 0$, $a \neq 0$ have **one zero**.

Quadratic equations $ax^2 + bx + c = 0$, $a \neq 0$ have **two zeros**.

Cases: **Repeated solution** $x = r, r$; r a real number.

$$x^2 + 8x + 16 = (x + 4)^2, \text{ zeros } x = -4 \text{ \& } -4$$

Distinct solutions $x = r, s$; r & s real numbers $r \neq s$

$$x^2 - 5x - 24 = (x - 8)(x + 3); \text{ zeros } x = 8 \text{ \& } -3$$

Complex solutions $x = s + ti, s - ti$; s & t real, $i = \sqrt{-1}$

$$x^2 - 4x + 13 = (x - (2 + 3i))(x - (2 - 3i)); \text{ zeros } x = 2 + 3i, 2 - 3i$$

Given a quadratic equation $p(x) = ax^2 + bx + c = 0$, $a \neq 0$.

Describe the graph of $y = p(x)$ in each of the following cases:

(1) Repeated zeros

(2) Distinct real zeros

(3) Complex zeros

Polynomial facts:

The **domain** of every polynomial is all real numbers unless you specifically restrict it to some interval.

The **graph** of $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is continuous.

A polynomial of degree n has n zeros some of which can be repeated and some may be complex numbers. (This repeats what we learned about x-intercepts.)

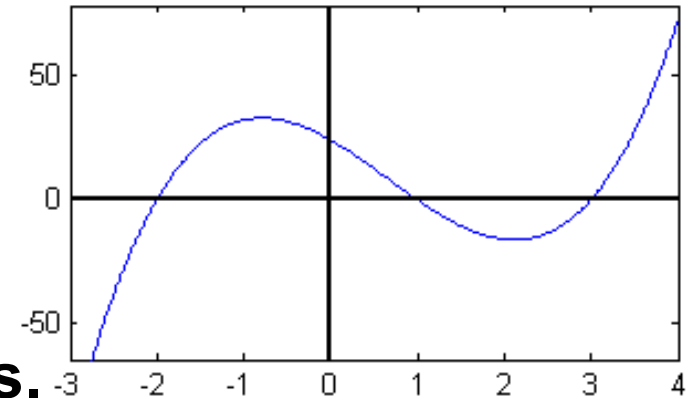
If you know all the zeros of a polynomial then you can “almost” write an equation for it.

Example: A cubic polynomial has exactly 3 zeros (some may be complex).

$$p(x) = 4x^3 - 8x^2 - 20x + 24$$

has zeros $x = 1, -2,$ and $3.$

Note the graph has 3 x-intercepts.

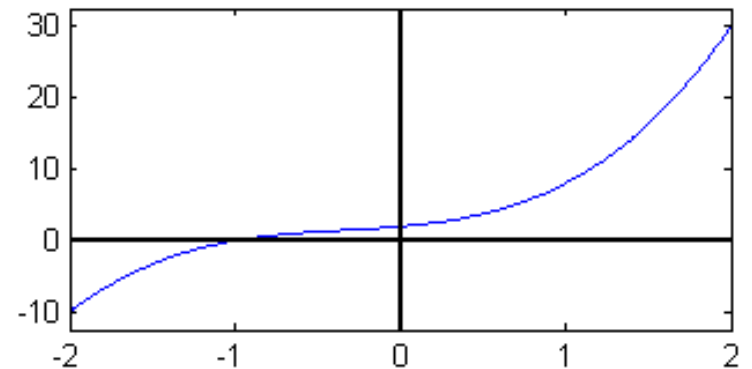


$$p(x) = 2x^3 + 2x^2 + 2x + 2$$

has zeros $x = -1, i, -i$

where $i = \sqrt{-1}$

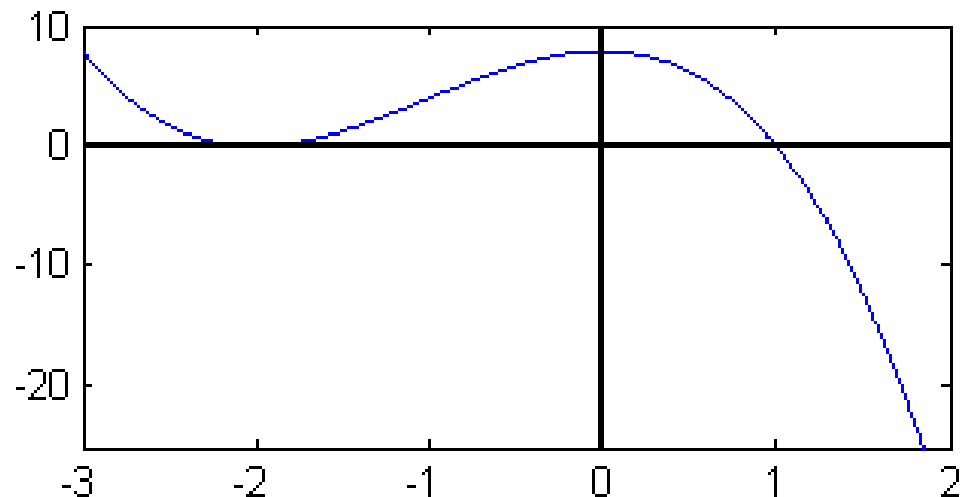
Note the graph has 1 x-intercept.



Example continued:

$$p(x) = -2x^3 - 6x^2 + 8$$

has zeros $-2, -2, 1$



Notice that the graph crosses the x-axis at $x = 1$ while at $x = -2$, it just touches the x-axis. (It does not cross it.)

Since -2 is a zero twice, we say it is “**repeated**” or it is called a “**multiple**” zero.

Multiplicity of a zero.

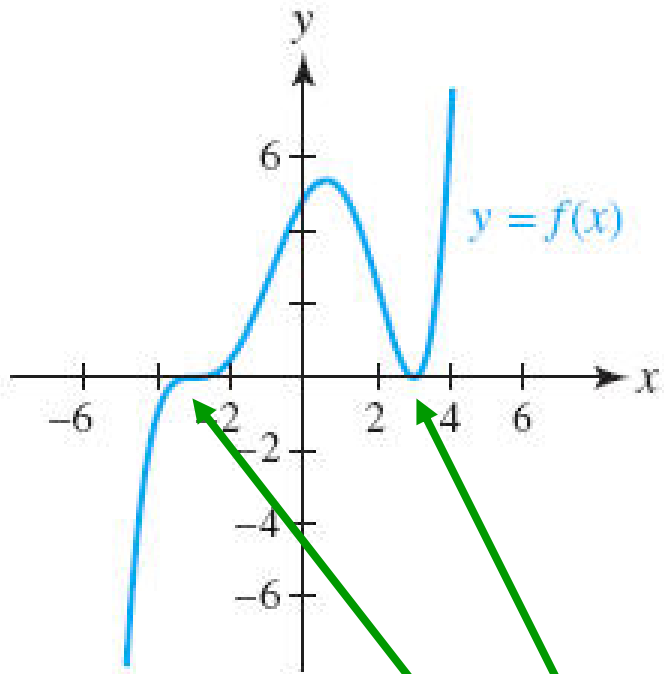
If $x = k$ is a zero of polynomial $f(x)$ more than once, then $x = k$ is called a **multiple zero**.

This means that $x - k$ is a factor of $f(x)$ more than once.

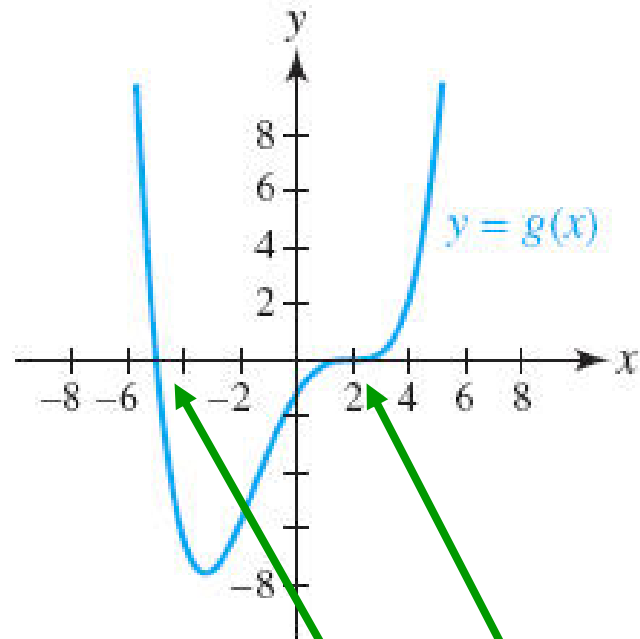
The number of times that k is a zero of polynomial $f(x)$ is called the **multiplicity** of the zero.

Example: The polynomial $f(x) = 0.02(x + 3)^3(x - 3)^2$
has zeros -3 and 3 with multiplicities **3**
and **2**, respectively.

Behavior of the graph: At a zero of **even multiplicity** the graph **does not cross the x-axis**, while at zero of **odd multiplicity** it **does cross the x-axis**.



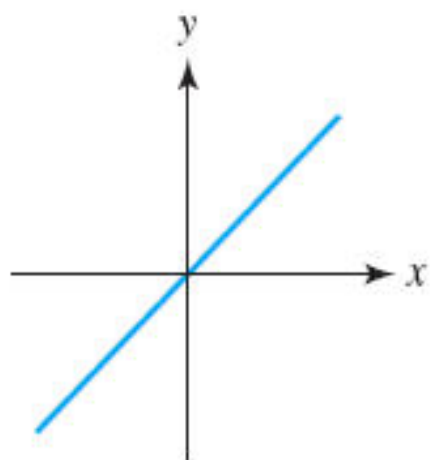
$$\underline{f(x) = 0.02(x + 3)^3(x - 3)^2}$$



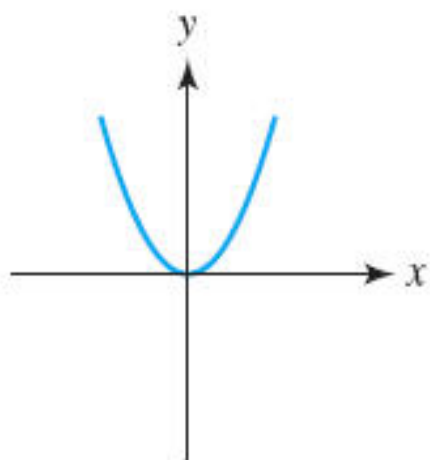
$$g(x) = 0.03(x + 5)(x - 2)^3$$

**$x = -5$ is a zero of multiplicity 1,
while $x = 2$ has multiplicity 3**

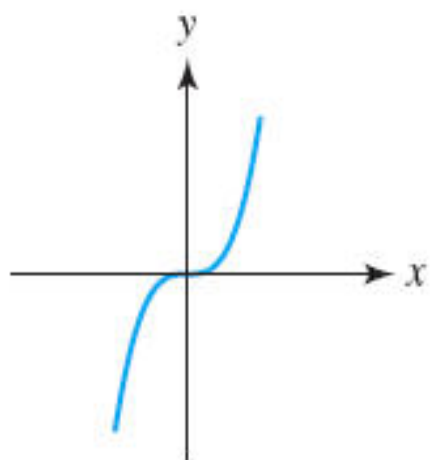
**Note the different behaviors for the zeros of
odd multiplicity and those of even multiplicity.**



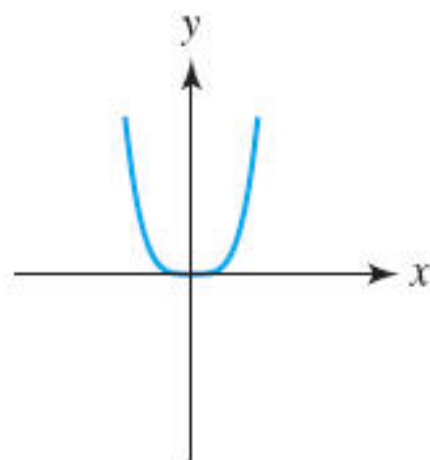
Multiplicity 1 (odd)



Multiplicity 2 (even)



Multiplicity 3 (odd)



Multiplicity 4 (even)

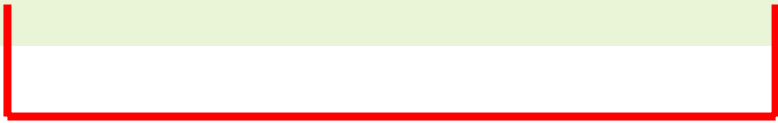
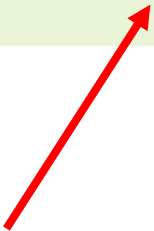


COMPLETE FACTORED FORM

Suppose a polynomial

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

has n real zeros $c_1, c_2, c_3, \dots, c_n$, where distinct zeros are listed as many times as their multiplicities. Then $f(x)$ can be written in **complete factored form** as


$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdots (x - c_n).$$


**Leading
coefficient**

The factors.

**So if c_1 is a zero, then $x - c_1$ is
a factor.**

**Note the /
“minus” sign.**



Example:

Use the graph of

$f(x) = x^3 - x^2 - 9x + 9$ and the factor theorem to list the factors of $f(x)$.

Solution

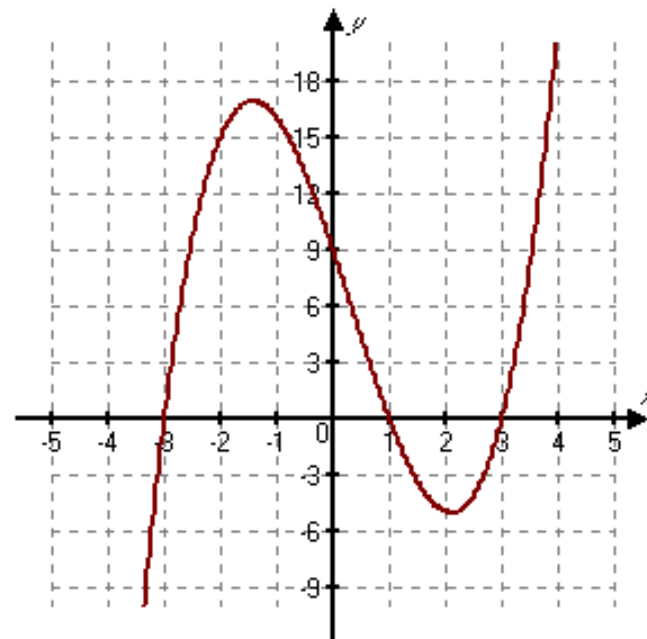
The graph shows that the zeros or x -intercepts of f are -3 , 1 , and 3 .

Since $f(-3) = 0$, the factor theorem states that $(x + 3)$ is a factor, and $f(1) = 0$

implies that $(x - 1)$ is a factor and

$f(3) = 0$ implies $(x - 3)$ is a factor.

Thus the factors are
 $(x + 3)(x - 1)$, and $(x - 3)$.



Example:

Write the complete factorization for the polynomial $6x^3 + 19x^2 + 2x - 3$ with zeros -3 , $-1/2$ and $1/3$.


Solution

Leading coefficient is 6

Zeros are -3 , $-1/2$ and $1/3$

Determine the factors: $(x - -3)$ $(x - -1/2)$ $(x - 1/3)$

The complete factorization:

$$f(x) = 6(x + 3) \left(x + \frac{1}{2}\right) \left(x - \frac{1}{3}\right)$$


Example:

Write the complete factorization for the polynomial $-2x^4 + 2x^3 + 34x^2 - 42x - 72$ given zeros 3, 3, -1 and -4 .

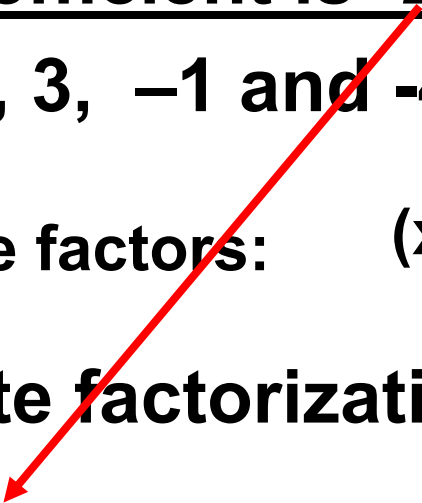
Solution

Leading coefficient is -2

Zeros are 3, 3, -1 and -4

Determine the factors: $(x - 3)^2 (x + 1) (x + 4)$

The complete factorization:

$$f(x) = -2(x - 3)^2 (x + 1) (x + 4)$$


Example:

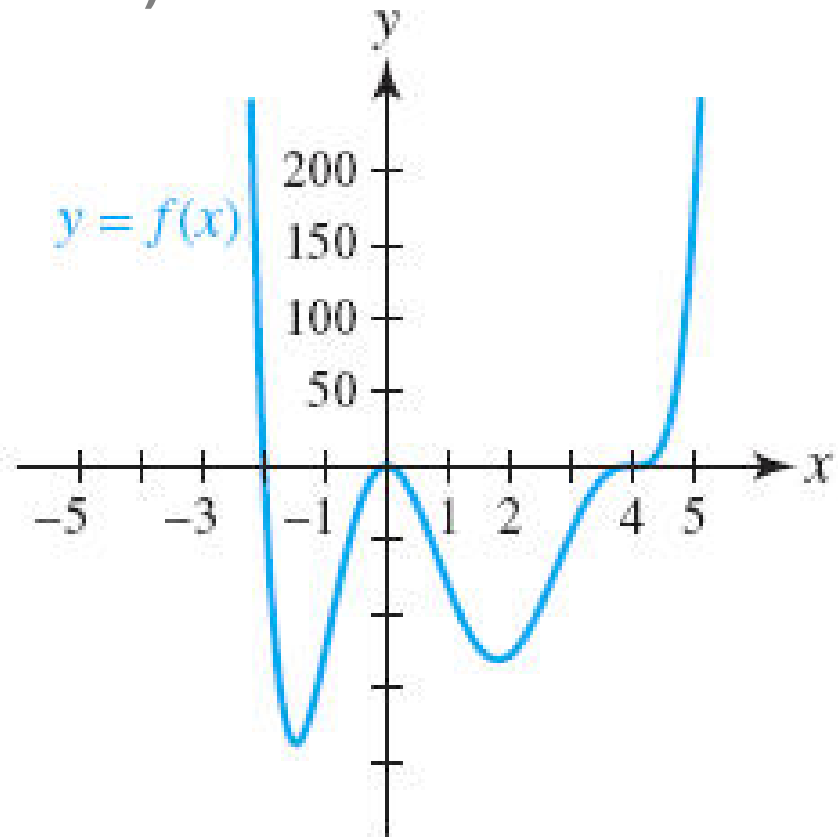
This is the graph of a polynomial of degree 6.
Construct its equation in complete factored form.
(Assume leading coefficient is 1.)

What are the zeros?

Zeros are: $x = -2, 0, 4$

Which are zeros of odd multiplicity and which are even multiplicity?

Hint: look at the way the curve crosses the x-axis.



$$f(x) = x^2 (x + 2)(x - 4)^3$$

◆ MAKING CONNECTIONS

x -Intercepts, Zeros, and Factors Let $f(x)$ be a polynomial with degree 1 or more. The following statements are equivalent.

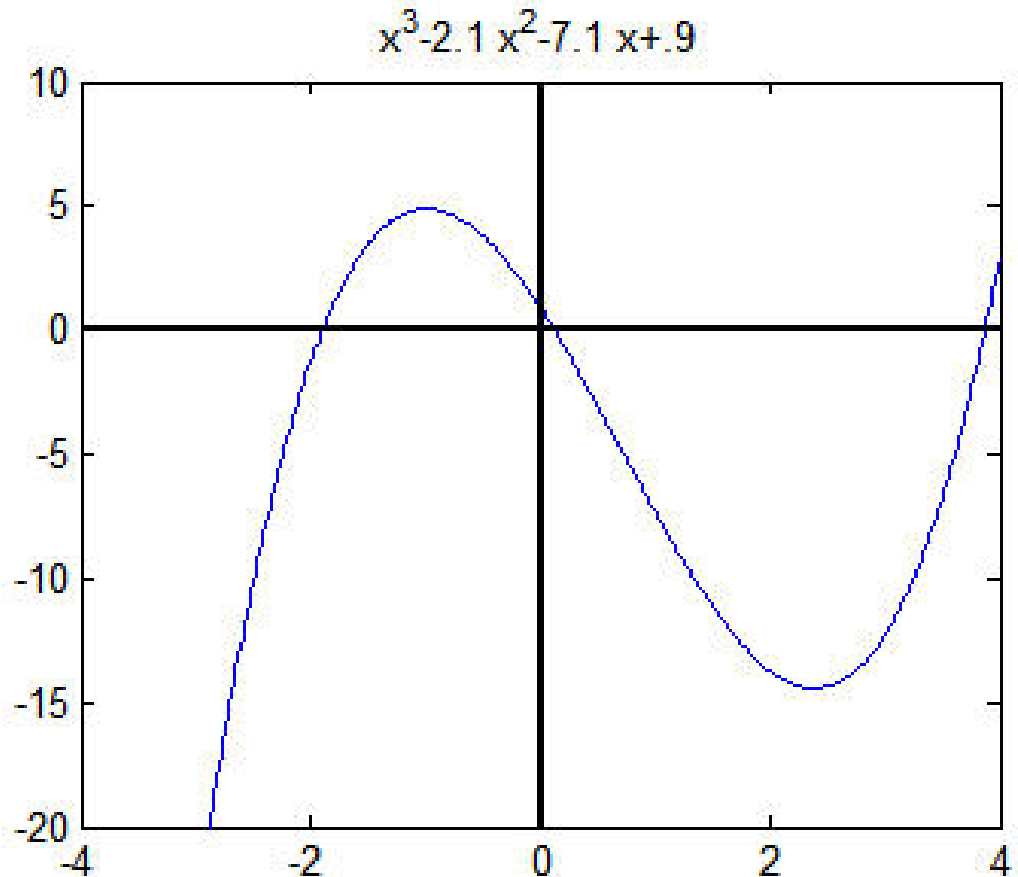
1. The graph of $y = f(x)$ has x -intercept k .
2. A zero of $f(x)$ is k . That is, $f(k) = 0$.
3. A factor of $f(x)$ is $(x - k)$.

Example:

Solve the equation $x^3 - 2.1x^2 - 7.1x + 0.9 = 0$ graphically.

Solution

Since there are three x -intercepts the equation has three real solutions.



Grapher/Tracer
with Grid.

$$x^3 - 2.1x^2 - 7.1x + 0.9 = 0$$

Domain [-4, 4], Range [-20, 10]

Approximate solutions

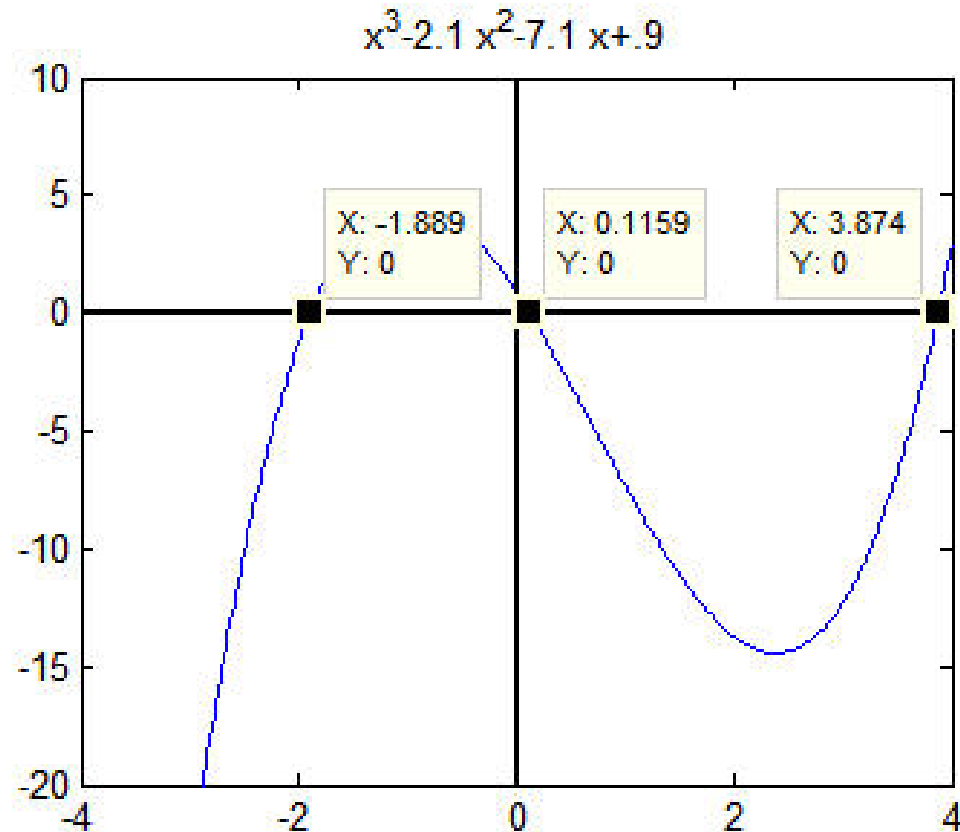
$x \approx .012, -1.89, \text{ and } 3.87$

Better numerical
approximations

3.8731

-1.8957

0.12258

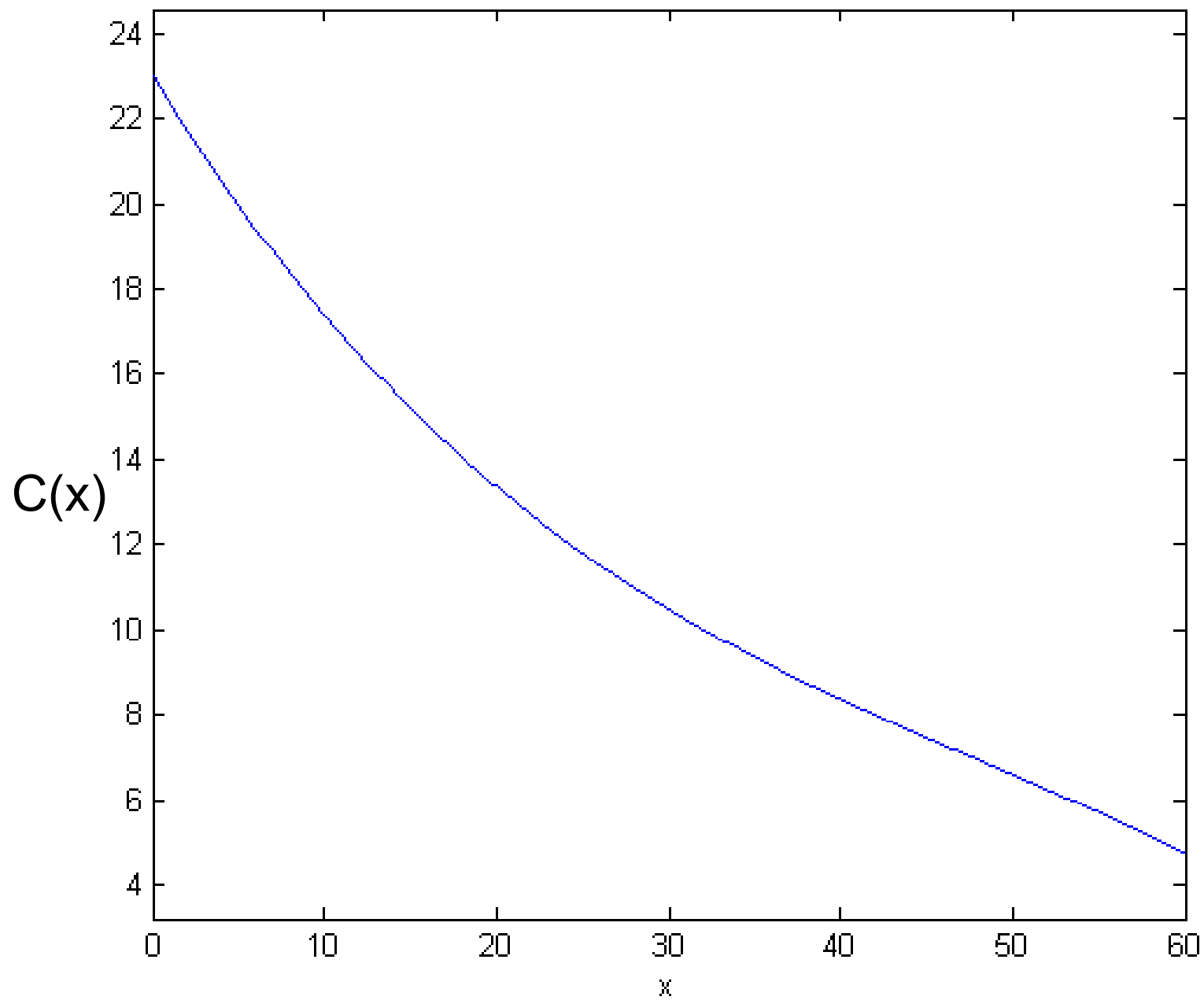


Example: High concentrations of copper in streams can be harmful to aquatic life. Concentration above 10 are lethal to fresh water mussels. A particular study collected data and modeled the copper concentration $C(x)$ in a stream where x is the distance down stream from the entry point of the copper contaminant. The equation for the model is

$$C(x) = -0.000068x^3 + 0.0099x^2 - 0.653x + 23$$

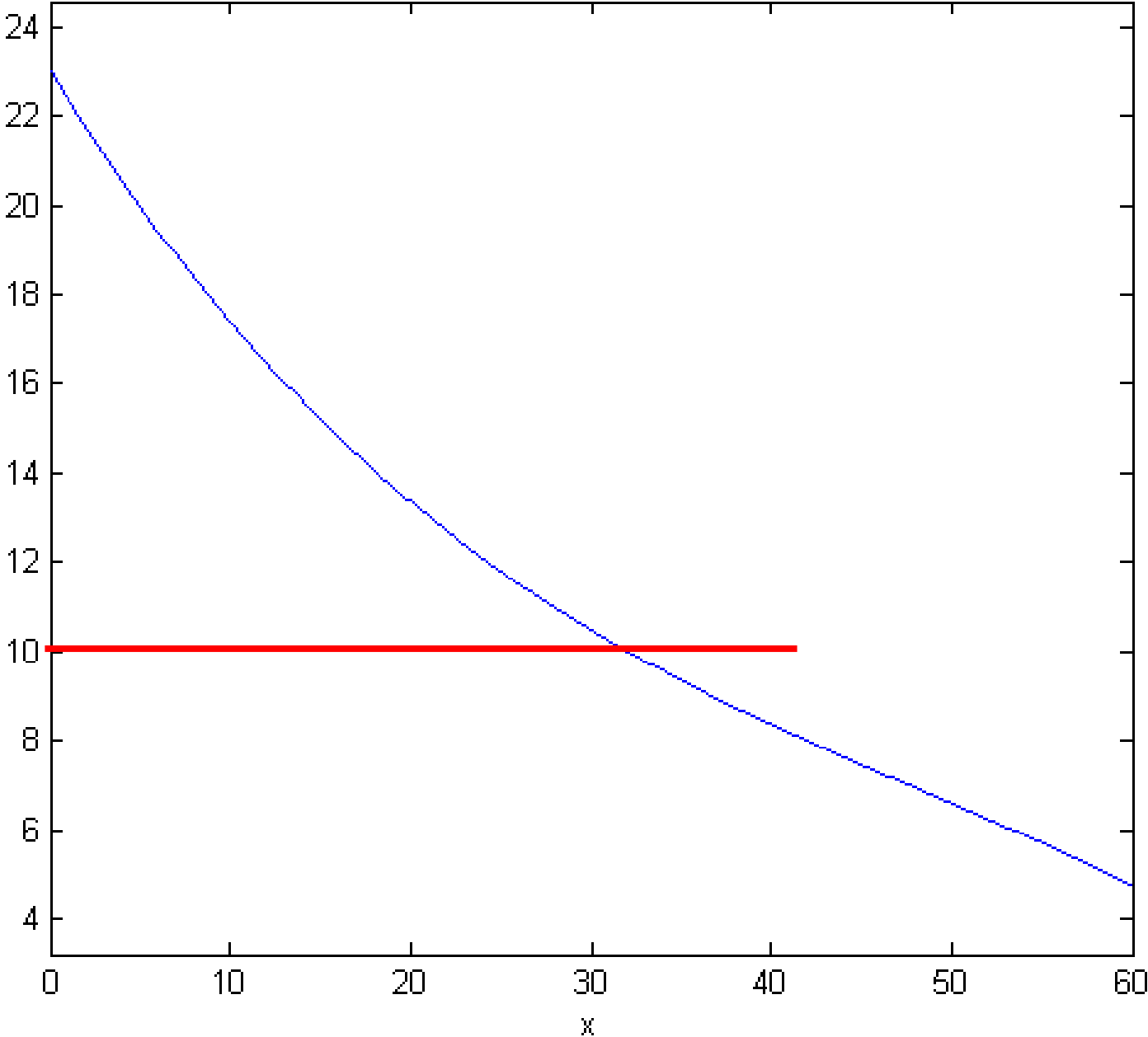
Explain how to determine the part of the stream that would be lethal to fresh water mussels.

$$-.000068 x^3 + .0099 x^2 - .653 x + 23$$



$$-.000068 x^3 + .0099 x^2 - .653 x + 23$$

C(x)



Example:

The cubic polynomial

$$f(x) = -0.184x^3 + 1.45x^2 + 10.7x - 27.9$$

models the monthly average temperature at Trout Lake, Canada, in degrees Fahrenheit, where $x = 1$ corresponds to January and $x = 12$ represents December.

Interpret the zeros of $f(x)$.

**Grapher/Tracer
with Grid.**

Use a graph window
 x in $[0, 12]$, y in $[-20, 80]$

$$f(x) = -0.184x^3 + 1.45x^2 + 10.7x - 27.9$$

