4.2

Polynomial Functions and Models

- Understand the graphs of polynomial functions.
- Evaluate and graph piecewise-defined functions

Graphs of Polynomial Functions

A polynomial function f of **degree** n can be expressed as

$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$, where each **coefficient** a_k is a real number, $a_n \neq 0$, and n is a nonnegative integer.

A **turning point** occurs whenever the graph of a polynomial function changes from increasing to decreasing or from decreasing to increasing. Turning points are associated with "hills" or "valleys" on a graph.

We want to use basic information about a polynomial to predict properties of its graph.

The basic information is:

- **The degree**
- **The sign of its leading coefficient**

The properties of the graph that are of interest are:

- **Maximum number of x-intercepts that are possible.**
- **Maximum number of turning points that are possible.**
- **Behavior of the graph as x becomes very negative and the behavior as x becomes very positive. Behavior at the "ends".**

$x \to \infty$ $\qquad$ $x \to -\infty$

Polynomial Behavior: x-intercepts & turning points

The best we can do is predict the maximum number of intercepts and turning points based on the degree of the polynomial.

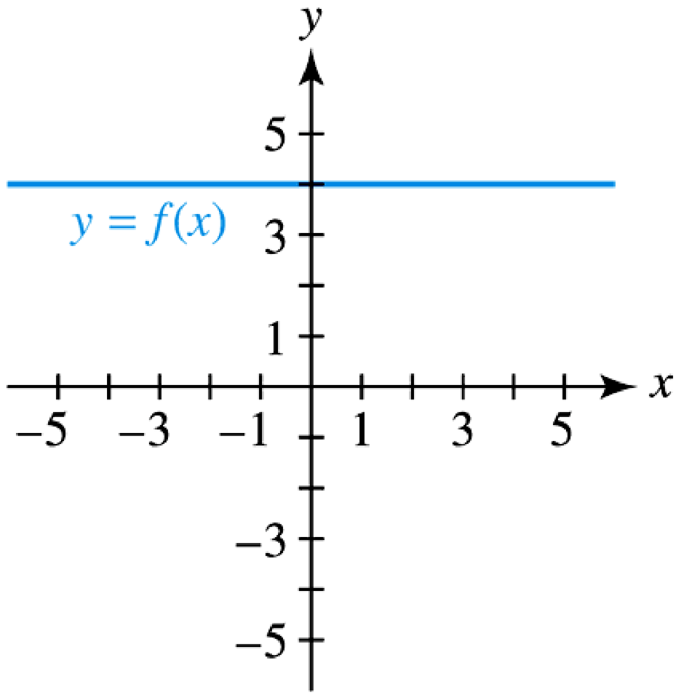
Make a chart like this on a piece of paper.

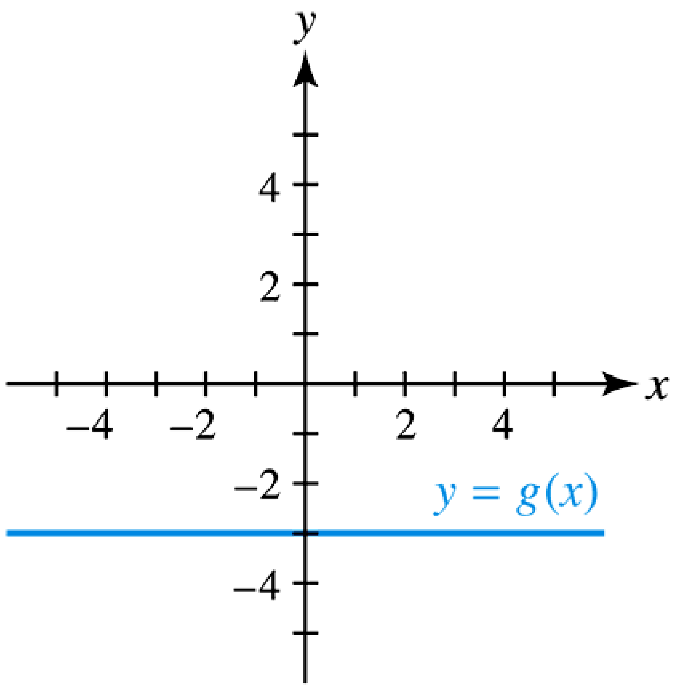
Degree	Max. number of x-intercepts	Max Number of turning points
1		
2		
3		
4		
5		
6		

Constant Polynomial Function y = f(x) = k

Has no *x*-intercepts or turning points. (Except f(x) = 0 which has infinitely many x-intercepts.)

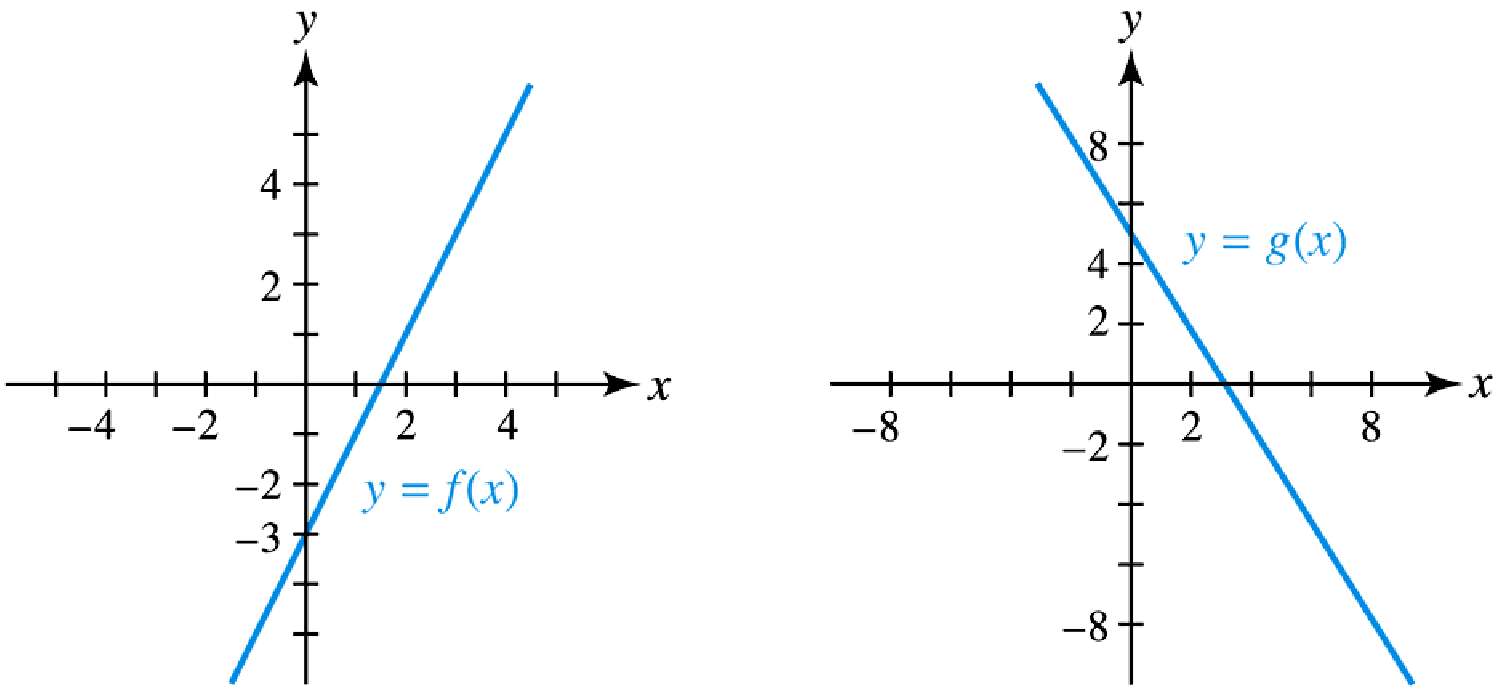
Degree 0.





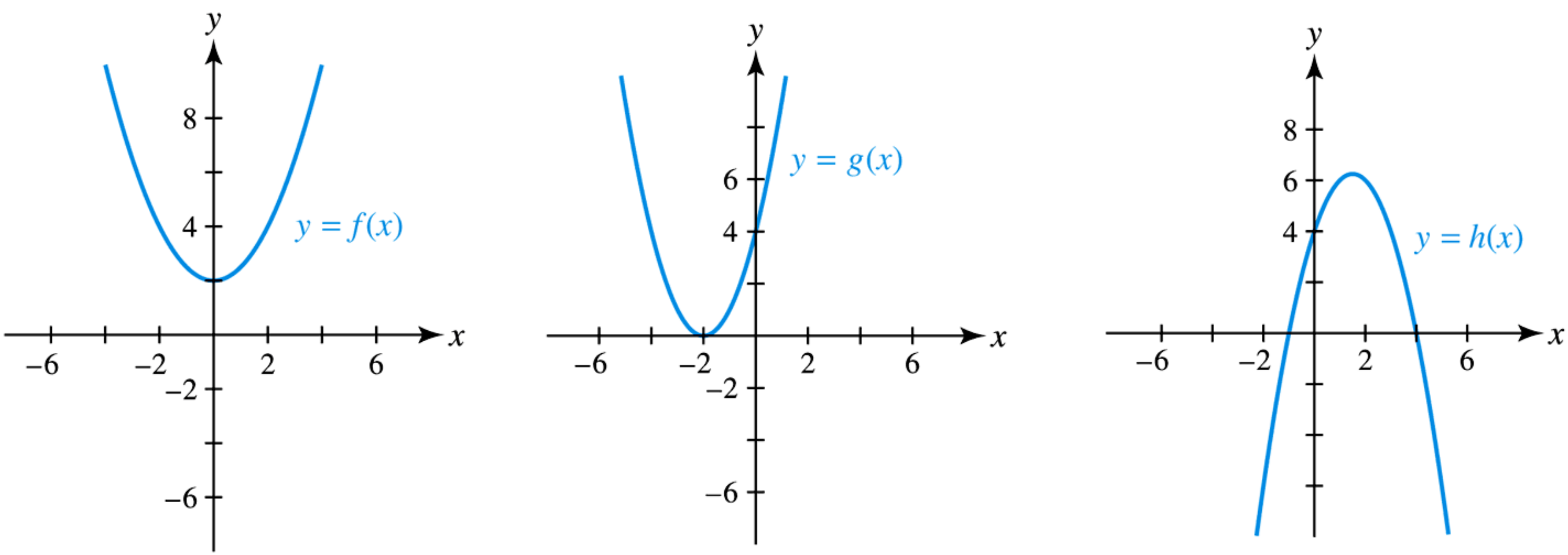
Linear Polynomial Function $y = f(x) = mx + b$, $m \neq 0$

Degree 1 and **one** ***x*-intercept** and **no** **turning points.**



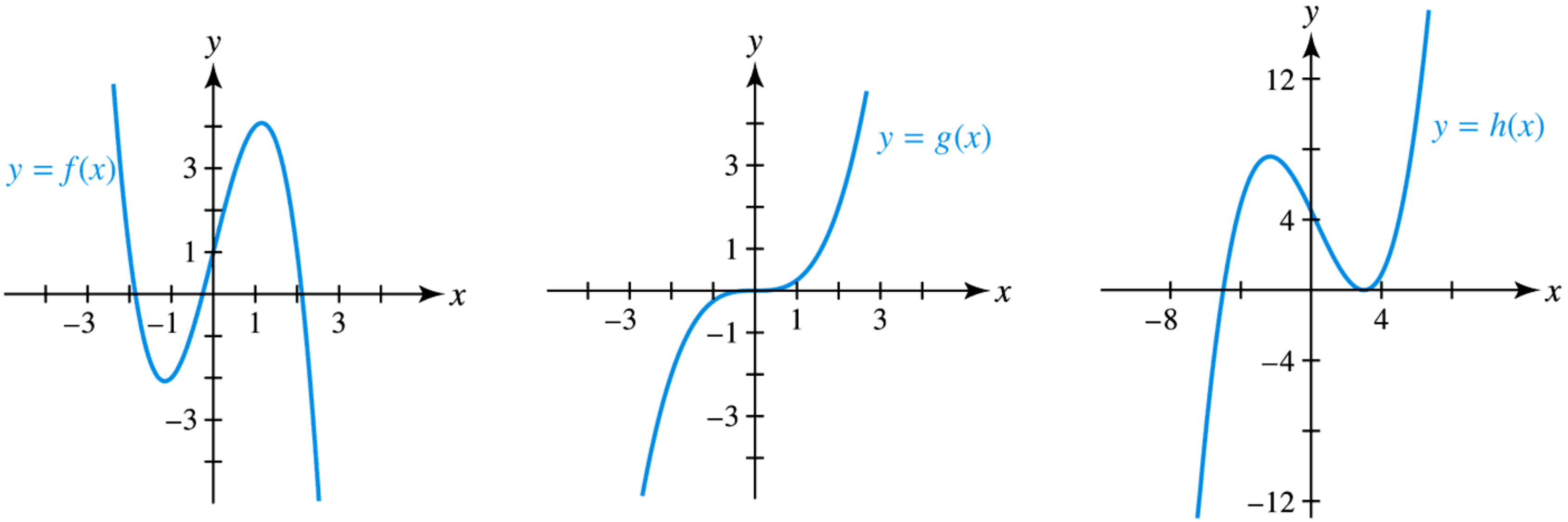
Quadratic Polynomial Functions
$y = f(x) = ax^2 + b + c$

Degree 2, parabola that opens up or down. Can have zero, one or two *x*-intercepts. Has exactly one turning point, which is also the vertex.



Cubic Polynomial Functions
$y = ax^3 + bx^2 + cx + d$

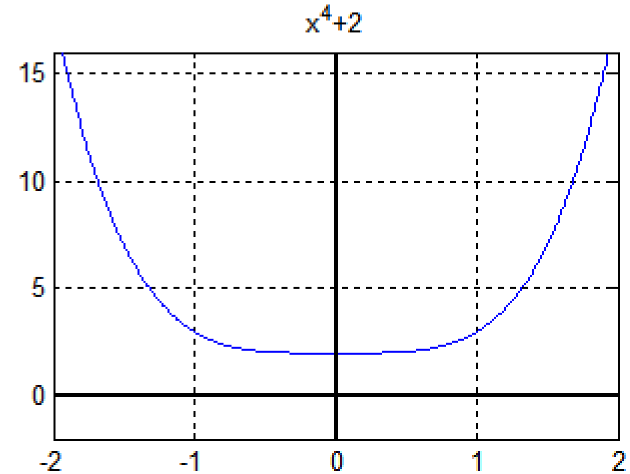
Degree 3, can have **zero or two turning points** and **one, two or three x-intercepts**.

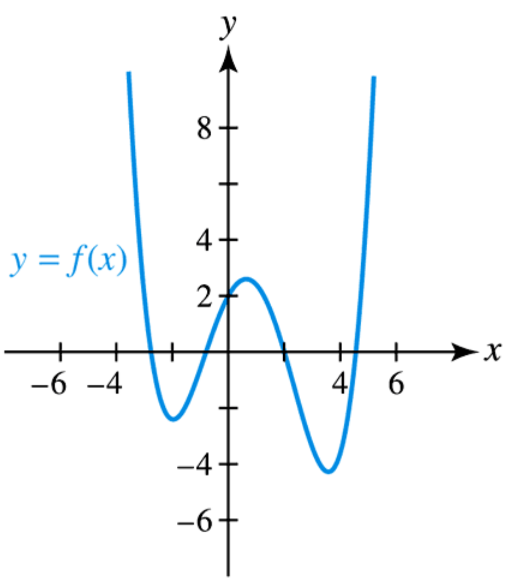


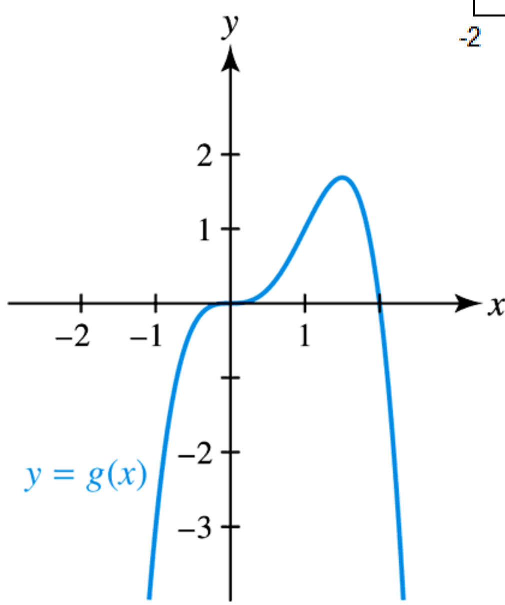
Quartic Polynomial Functions

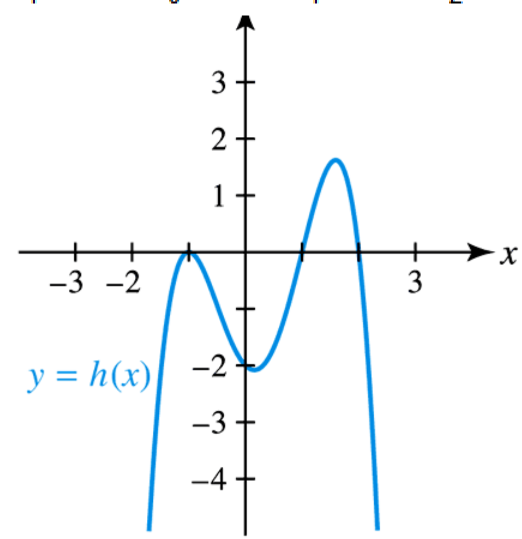
$y = f(x) = ax^4 + bx^3 + cx^2 + dx + e$

Degree 4, can have up to four *x*-intercepts and up to three turning points.



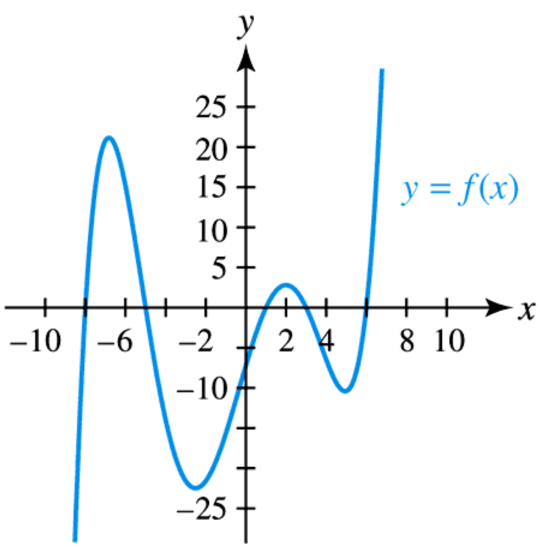


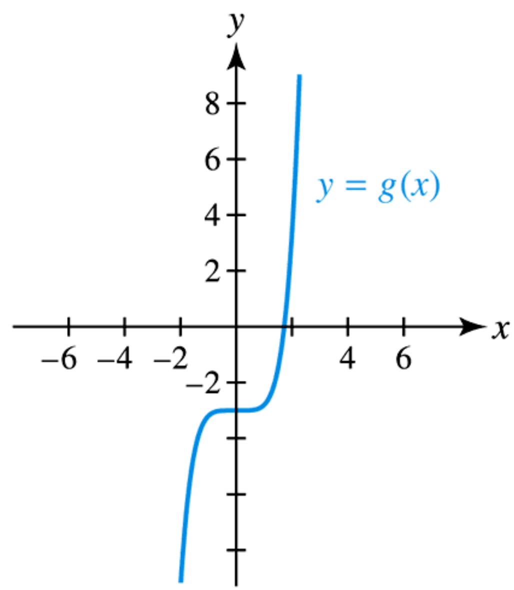


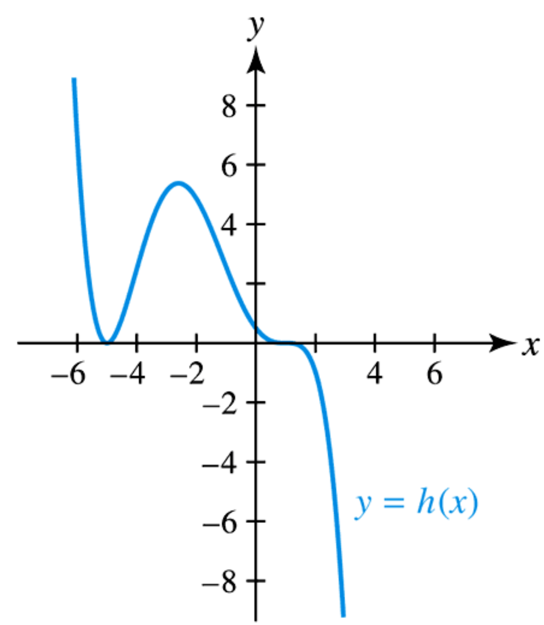


Quintic Polynomial Functions
$y = g(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

Degree 5, may have up to ????? *x*-intercepts and up to ???? turning points.



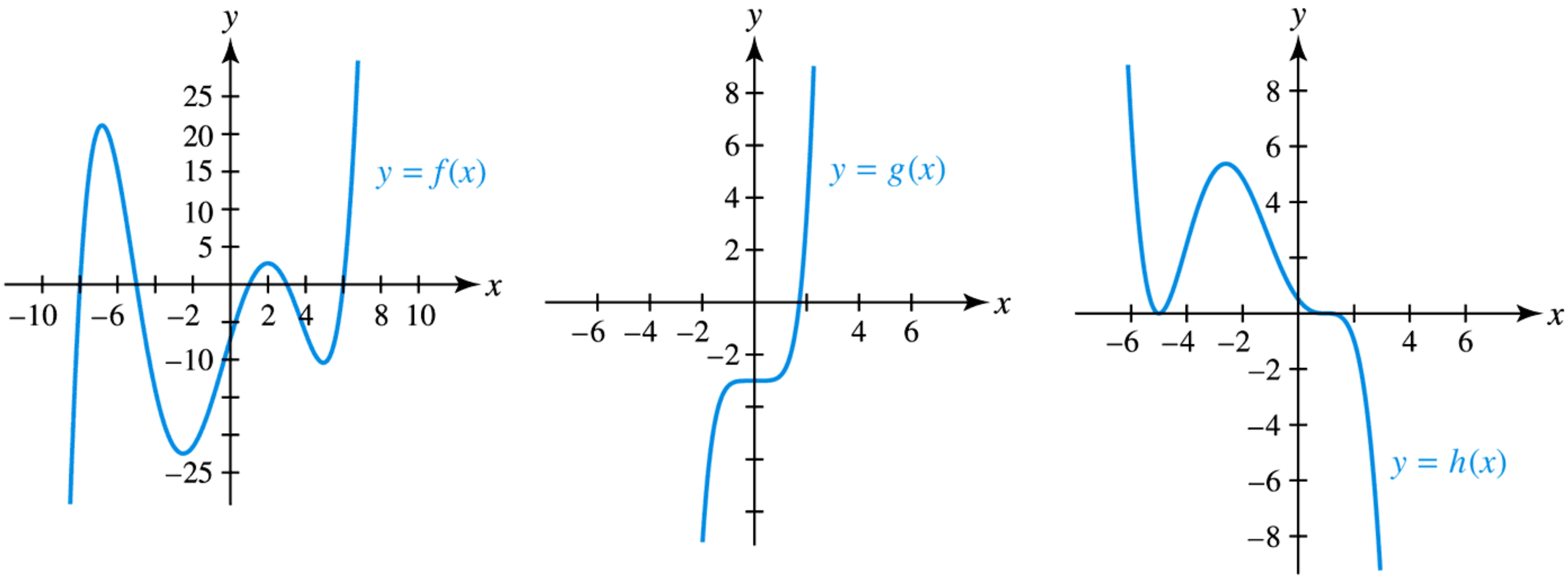




Quintic Polynomial Functions

$y = g(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

Degree 5, may have up to five *x*-intercepts and up to four turning points.



Polynomial Behavior: x-intercepts & turning points

The best we can do is predict the maximum number of intercepts and turning points based on the degree of the polynomial.

Degree	Max. number of x-intercepts	Max Number of turning points
1	1	0
2	2	1
3	3	2
4	4	3
5	5	4
6	6	5

From the preceding cases we can "conjecture" a pattern for the following.

Given that y = f(x) is a polynomial of degree $n \geq 1$, then there are up to ______ x-intercepts and up to ______ turning points.

(Fill in the blanks.)

Example: **If y = f(x) is a polynomial of degree 8, then it has at most ______ x-intercepts and at most ______ turning points.**

Next we discuss how the degree and the sign of the leading coefficient tells the behavior of the graph of the polynomial near the "ends"; that is, as x gets very positive and as x gets very negative.

$x \to \infty$ $\quad\quad$ $x \to -\infty$

Let *f* be a polynomial of degree *n* like

$$f(x) = a_n x^n + \dots + a_2 x^2 + a_1 x + a_0$$

As x get very positive or x gets very negative the term $a_n x^n$ <u>"DOMINATES"</u> *the numerical values compared to all the other terms when f is evaluated at those x's. So the behavior of the graph at the ends is completely determined by "the leading term".*

<u>There are only two possible behaviors at the ends:</u>

1. **y = f(x) rises & keeps going up.**
2. **y = f(x) falls & keeps going down.**

End Point Behavior of Polynomials

The behavior as x →-∞ (left end) or as x →∞ (right end) is determined the "leading term".

Degree	Sign of leading coefficient	Left end behavior	Right end behavior
EVEN	Positive		
EVEN	Negative		
ODD	Positive		
ODD	Negative		

Let's consider an example:

$y = f(x) = 5x^4 - 9x^3 + 7x^2 - 121x + 89$

Leading term

Even degree & positive leading coefficient

How do the values of the leading term behave as x gets very positive? → Big positive values!

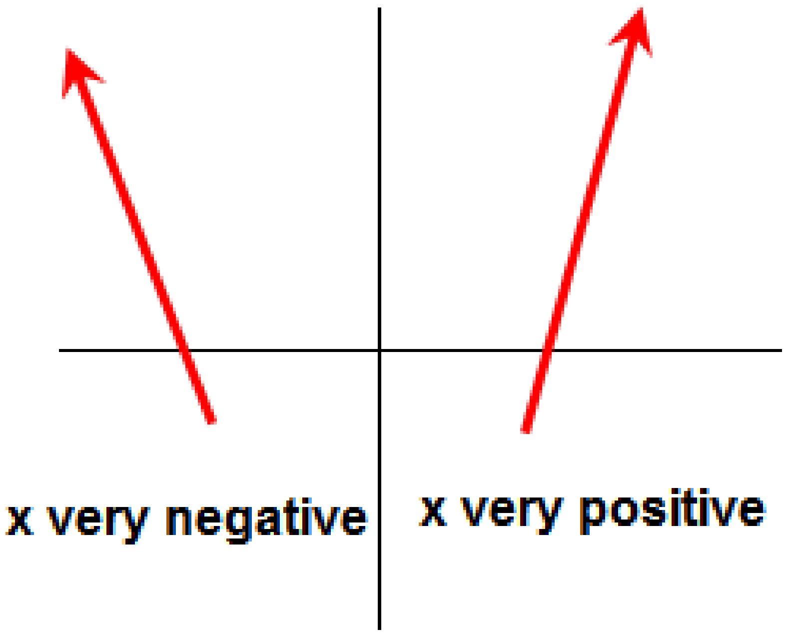
How do the values of the leading term behave as x gets very negative? → Big positive values!

What happens if the leading term were $-3x^4$?

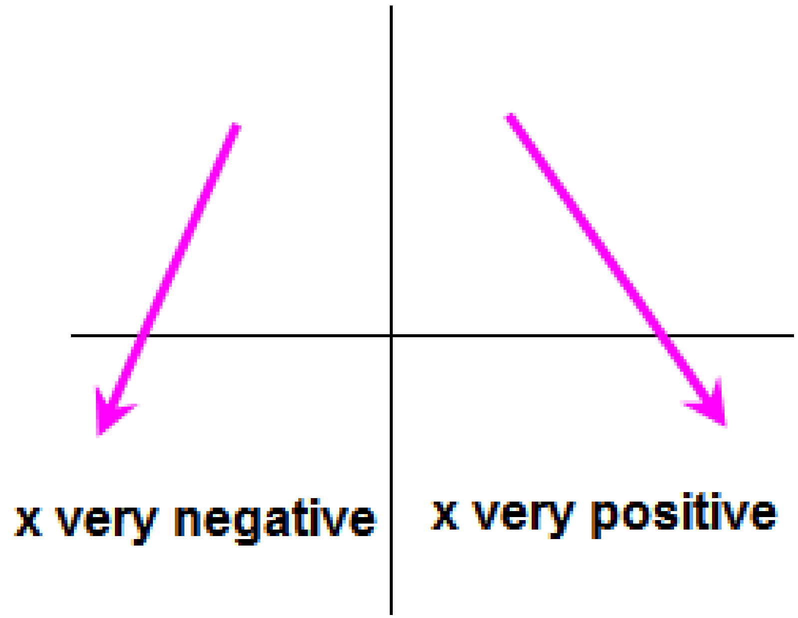
x gets very positive → Very negative values!

x gets very negative → Very negative values!

Case: Even degree & leading coefficient positive; "end behavior"



Case: Even degree & leading coefficient negative; "end behavior".



Another example:

$y = f(x) = 17x^5 - 9x^3 + 7x^2 - 121x + 89$

Leading term

Odd degree & positive leading coefficient

How do the values of the leading term behave as x gets very positive? → Big positive values!

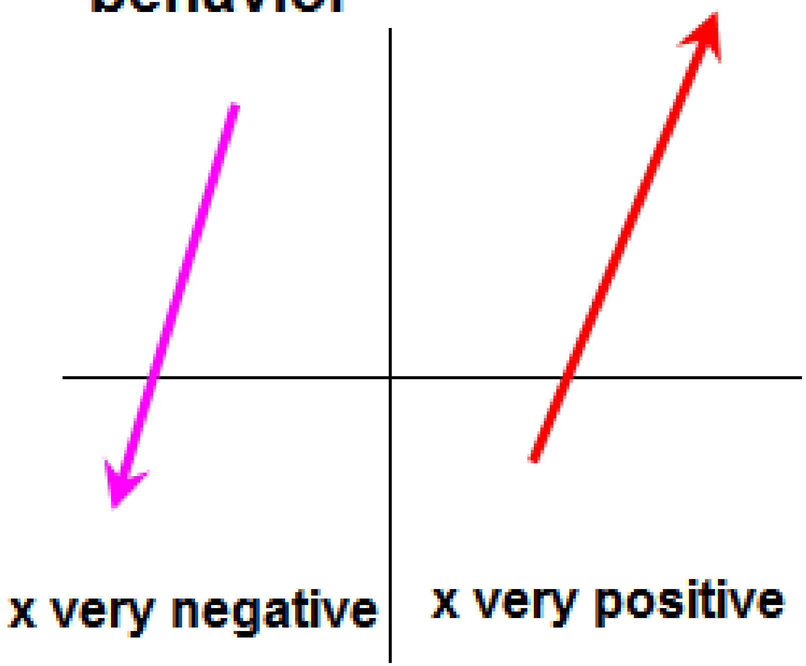
How do the values of the leading term behave as x gets very negative? → Very negative values.

What happens if the leading term was $-31x^5$?

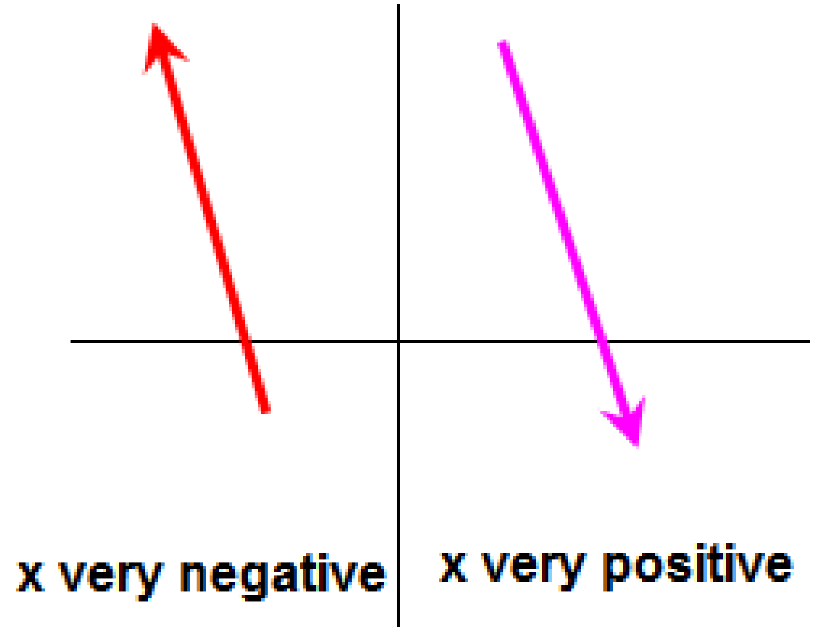
x gets very positive → Very negative values.

x gets very negative → Big positive values!

Case: Odd degree & leading coefficient positive; "end behavior"



Case: Odd degree & leading coefficient negative; "end behavior".



End Point Behavior of Polynomials

The behavior as x →-∞ (left end) or as x →∞ (right end) is determined the "leading term".

Degree	Sign of leading coefficient	Left end behavior	Right end behavior
EVEN	Positive	↑	↑
EVEN	Negative	↓	↓
ODD	Positive	↓	↑
ODD	Negative	↑	↓

Technical mathematical description of “End Behavior”

END BEHAVIOR OF POLYNOMIAL FUNCTIONS

Let f be a polynomial function with leading coefficient a and degree n.

1. $n \geq 2$ is even.

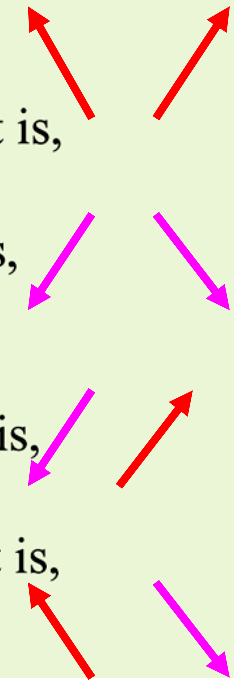
$a > 0$ implies the graph of f rises both to the left and to the right. That is, $f(x) \to \infty$ as $x \to \pm\infty$.

$a < 0$ implies the graph of f falls both to the left and to the right. That is, $f(x) \to -\infty$ as $x \to \pm\infty$.

2. $n \geq 1$ is odd.

$a > 0$ implies the graph of f falls to the left and rises to the right. That is, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

$a < 0$ implies the graph of f rises to the left and falls to the right. That is, $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.



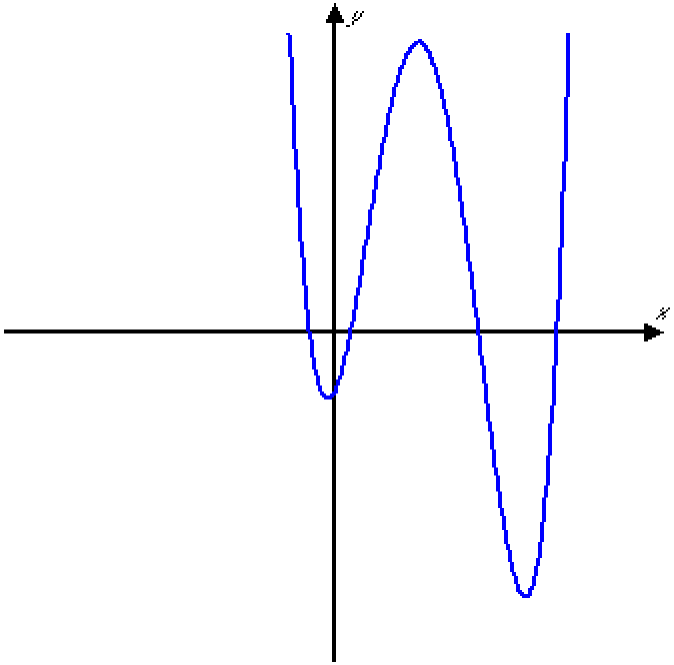
Example:

Use the graph of the polynomial function shown.

a) How many turning points and *x*-intercepts are there?

b) Is the leading coefficient *a* positive or negative? Is the degree odd or even?

c) Determine the minimum degree of *f*.

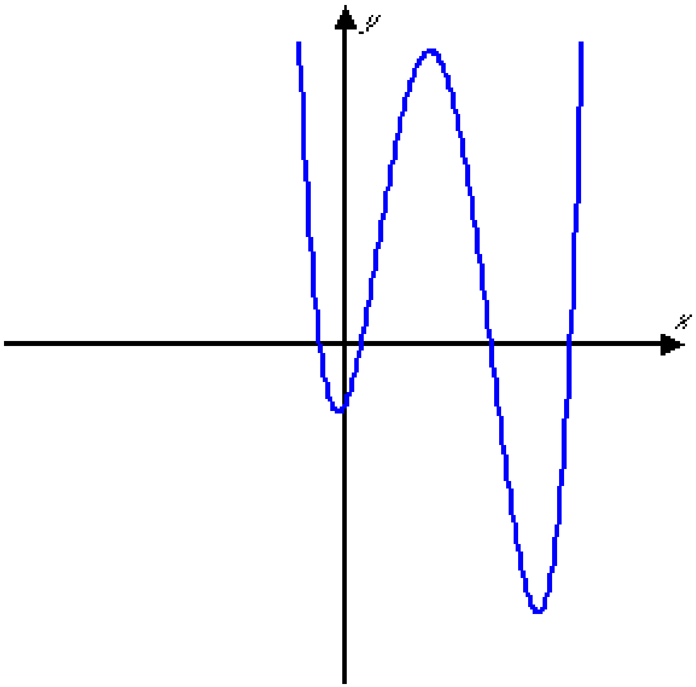


Solution

a) There are three turning points corresponding to the one "hill" and two "valleys". There appear to be 4 *x*-intercepts.

Solution continued

b) Is the leading coefficient *a* positive or negative? Is the degree odd or even?



The left side and the right side rise. Therefore, $a > 0$ and the polynomial function has even degree.

c) Determine the minimum degree of *f*.

The graph has three turning points. A polynomial of degree *n* can have at most $n - 1$ turning points. Therefore, *f* must be at least degree 4.

Example: (Use a calculator or graphing program.)

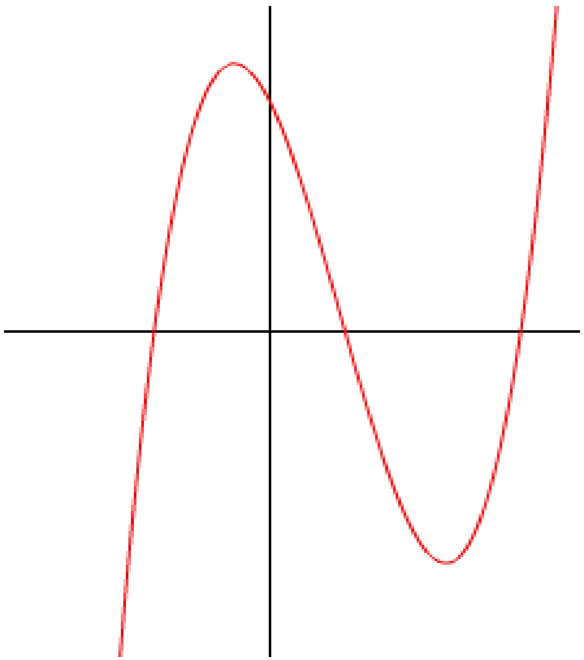
Graph $f(x) = 2x^3 – 5x^2 – 5x + 7$, and then complete the following.

a) Identify the x-intercepts.

b) Approximate the coordinates of any turning points to the nearest hundredth.

c) Use the turning points to approximate any local extrema.

Go to a Grapher/Tracer with Grid.



Book solution: The graph appears to intersect the x-axis at the points (–1.3, 0), (0.89, 0), and (2.9, 0).

Solution continued (keep using the grapher/tracer)

b) There are two turning points.

Book solution: From the graphs their coordinates are approximately (–0.40, 8.1) and (2.07, –7.04)

c) Local extrema.

Book solution: There is a local maximum of about 8.07 and a local minimum of about –7.04.

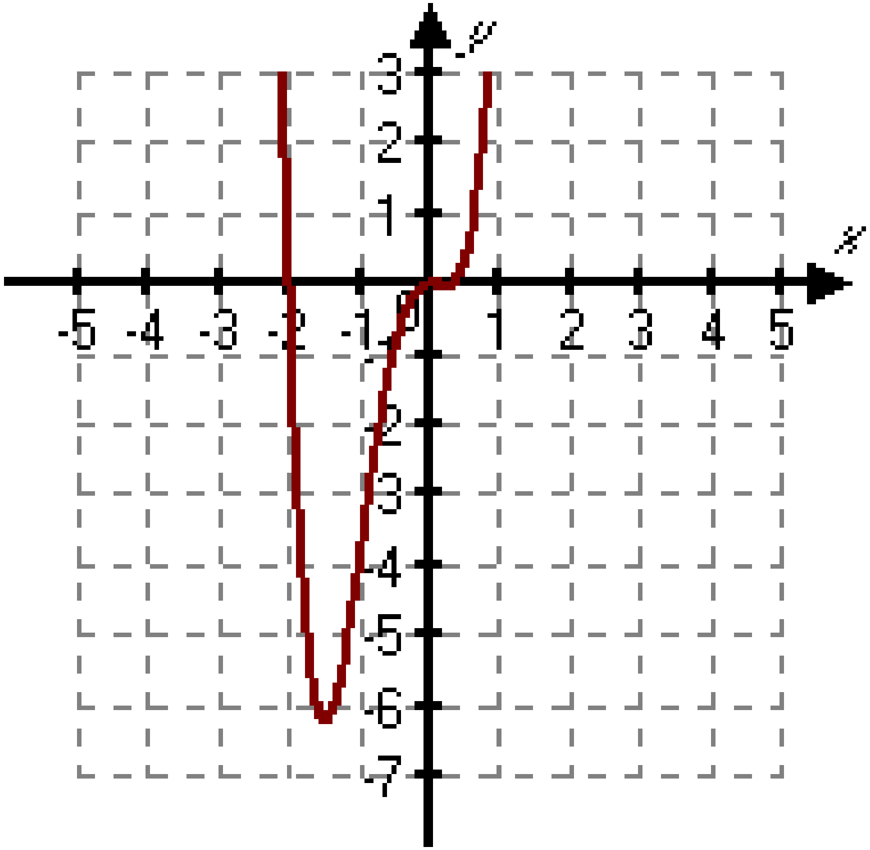
Example:

Let $f(x) = 3x^4 + 5x^3 - 2x^2$.

a) Give the degree and leading coefficient.

b) State the end behavior of the graph of *f*.

$$f(x) \to \infty \text{ as } x \to \pm\infty$$



Example:

Graph the polynomial y = f(x) in the standard calculator graphing window xmin = -10, xmax = 10, ymin = -10, ymax = 10.

$$y = f(x) = \tfrac{3}{16}x^4 - \tfrac{3}{2}x^2 + 5$$

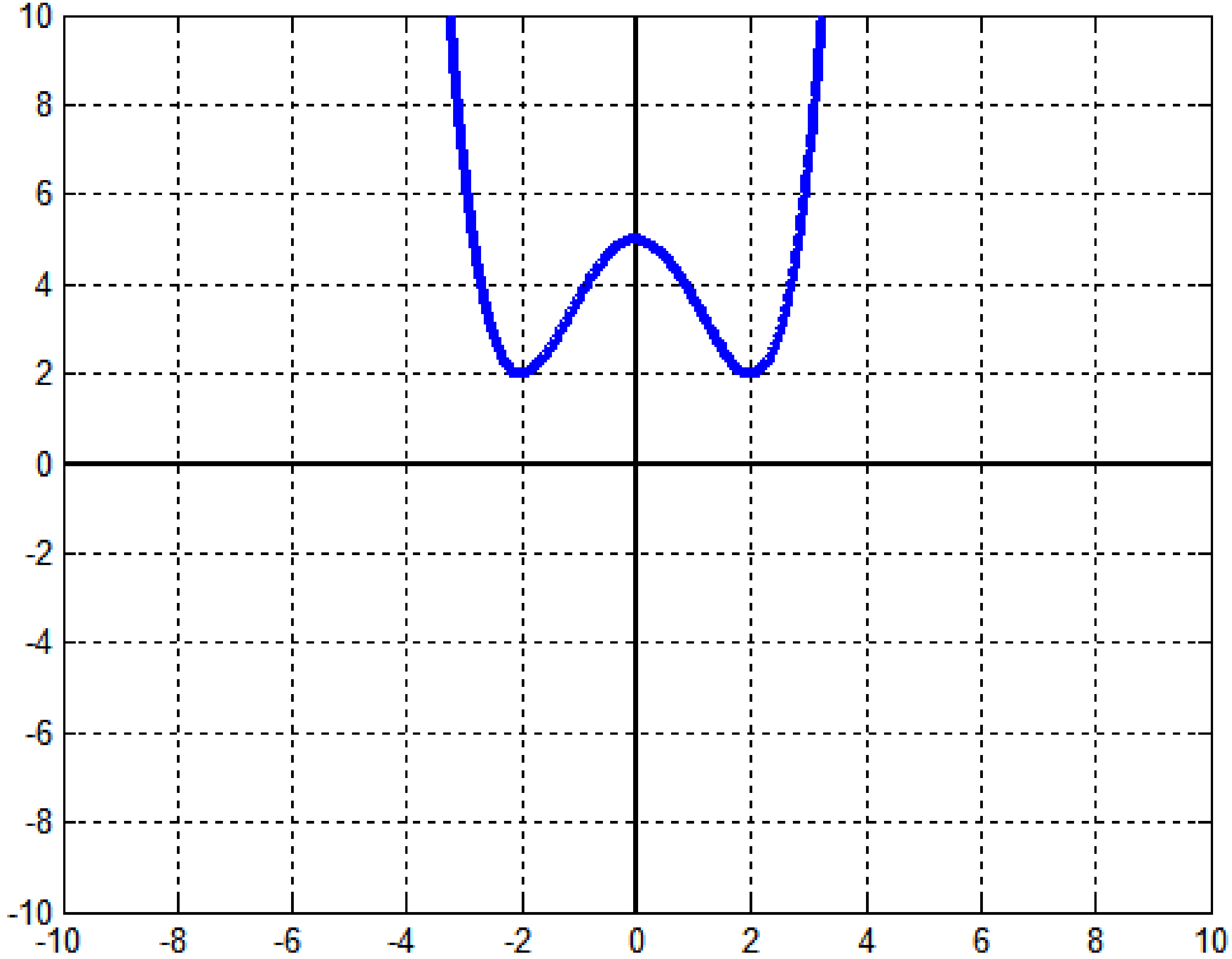
a) Find the turning points. (Give the answers as ordered pairs.)

b) Find the x-coordinate of any local minima.

c) Find the x-coordinate of any local maxima.

Go to a Grapher/Tracer with Grid.

Graph

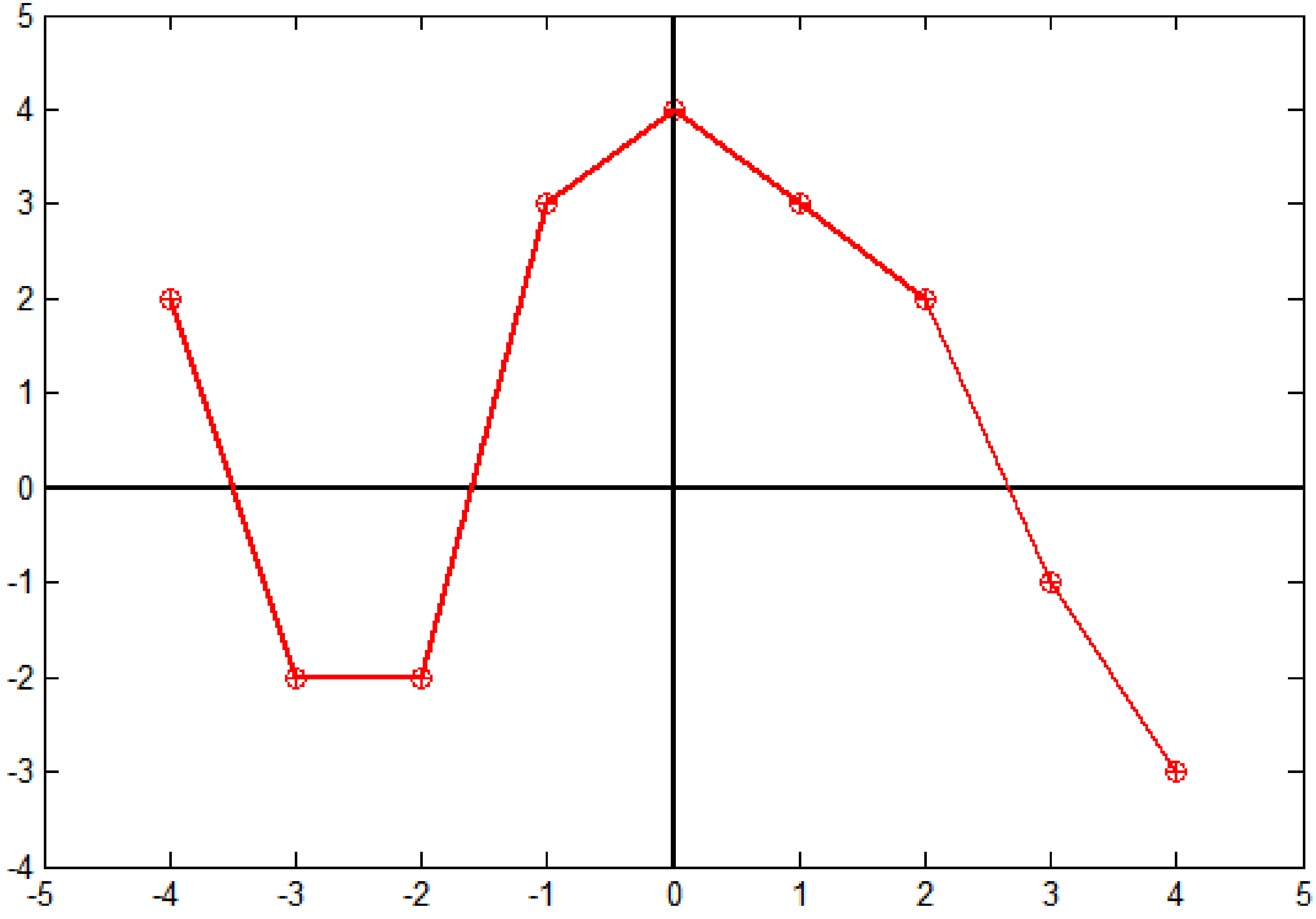


Example:

A data table has been generated by a linear, quadratic or cubic polynomial f(x). All zeros of f lie in the interval [-4, 4]. Make a graph of the data and then conjecture the degree of f.

x	-4	-3	-2	-1	0	1	2	3	4
f(x)	2	-2	-2	3	4	3	2	-1	-3

GRAPH



Piecewise-Defined Polynomial Functions

Example: **Evaluate $f(x)$ at –6, 0, and 4.**

$$f(x) = \begin{cases} -5x & \text{if } x < -5 \\ x^3 + 1 & \text{if } -4 < x \le 2 \\ 3 - x^2 & \text{if } x > 2 \end{cases}$$

Solution

To evaluate $f(-6)$ we use the formula $-5x$ because –6 is < –5. $f(-6) = -5(-6) = 30$

Similarly, $f(0) = x^3 + 1 = (0)^3 + 1 = 1$

And we have $f(4) = 3 - x^2 = 3 - (4)^2 = -13$

Example:

Complete the following.

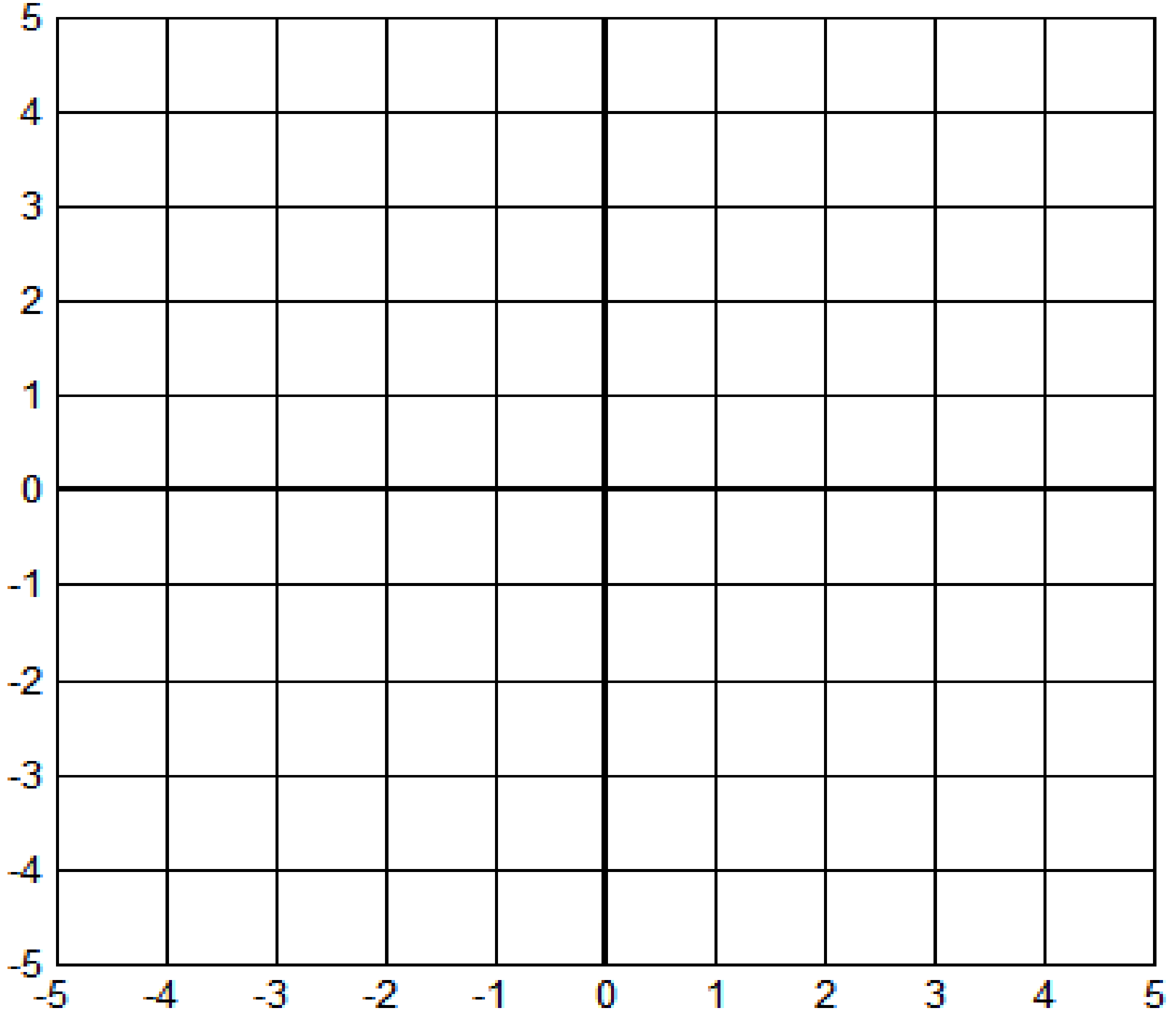
a) Evaluate *f*(1).

b) Sketch the graph of *f*.

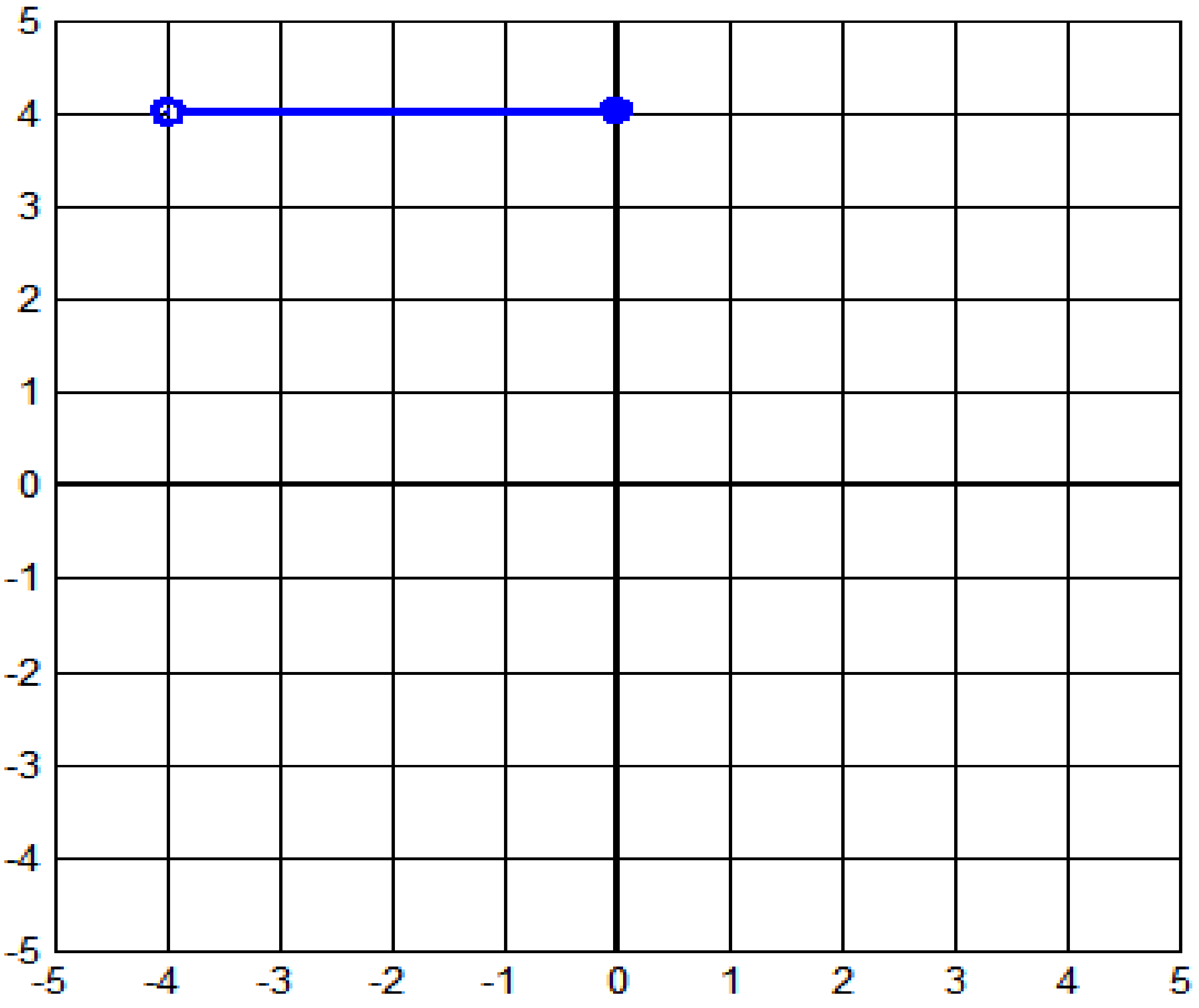
c) Determine if *f* is continuous on its domain.

$$f(x) = \begin{cases} 4 & \text{if } -4 < x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \\ 2x - 6 & \text{if } 2 < x < 4 \end{cases}$$

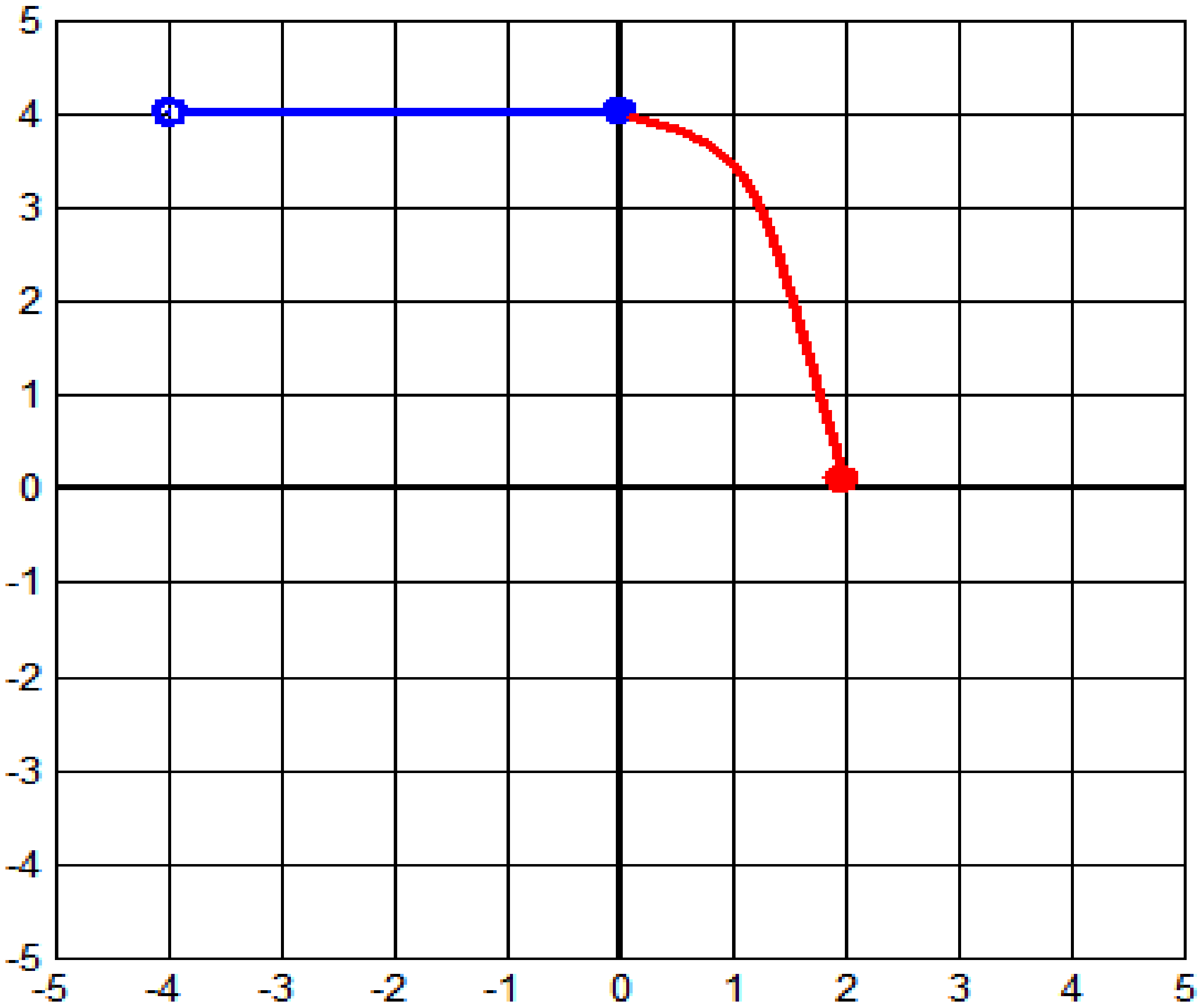
$$f(x)=\begin{cases}4 & \text{if } -4<x\le 0\\ 4-x^2 & \text{if } 0<x\le 2\\ 2x-6 & \text{if } 2<x<4\end{cases}$$



$$f(x) = \begin{cases} 4 & \text{if } -4 < x \leq 0 \\ 4 - x^2 & \text{if } 0 < x \leq 2 \\ 2x - 6 & \text{if } 2 < x < 4 \end{cases}$$



$$f(x)=\begin{cases}4 & \text{if } -4<x\le 0\\ 4-x^2 & \text{if } 0<x\le 2\\ 2x-6 & \text{if } 2<x<4\end{cases}$$



$$f(x)=\begin{cases}4 & \text{if } -4<x\le 0\\ 4-x^2 & \text{if } 0<x\le 2\\ 2x-6 & \text{if } 2<x<4\end{cases}$$