3.2

Quadratic Equations and Problem Solving

- Understand basic concepts about quadratic equations
- Use factoring, the square root property, and the quadratic formula to solve quadratic equations
- Understand the discriminant
- Solve problems involving quadratic equations

QUADRATIC EQUATION

A **quadratic equation** in one variable is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers with $a \neq 0$.

We need to "solve" quadratic equations $ax^2 + b + c = 0$ in a variety of situations. By solve we mean determine <u>all</u> the values of x, which when substituted into the equation give a "true" result.

<u>Example:</u> Does $x = 4$ solve $2x^2 + 2x - 12 = 0$?

$$\textbf{Compute: } 2(4)^2 + 2(4) - 12 = 32 + 8 - 12 = 40 - 12 = 28 \neq 0$$

Answer: NO **Is $x = -3$ a solution? Answer: YES**

When do we need to solve quadratics?

- **When graphing the parabola $y = ax^2 + bx + c$ the places the graph crosses the x-axis (the x-intercepts) are values of x that satisfy $ax^2 + bx + c = 0$.**

- **If we have a projectile problem modeled by the parabola $y = ax^2 + bx + c$ it will hit the ground at value of x that satisfies $ax^2 + bx + c = 0$.**

- **If we have a projectile problem modeled by the parabola $y = ax^2 + bx + c$ to determine the time x it reaches a height d we need to solve $ax^2 + bx + c = d$. To do that we must solve $ax^2 + bx + c – d = 0$.**

- **Other applications that use quadratic models: volumes of boxes, revenue computations, spread of disease, etc.**

Solving Quadratic Equations

We will consider three symbolic strategies in which quadratic equations can be solved.

- **Factoring (We will assume you can factor "reasonable" quadratics.)**
- **Square root property**
- **Quadratic formula**

"Reasonable" quadratics $ax^2 + bx + c = 0$ have integer coefficients where you can use the factors of the constant term c to "*help*" with the factoring.

FACTORING:

"Reasonable" quadratics $ax^2 + bx + c = 0$ have integer coefficients where you can use the factors of the constant term c to help with the factoring.

Examples:

$x^2 + x - 6 = 0$ → $(x - 2)(x + 3) = 0$

$2x^2 + 9x - 5 = 0$ → $(2x - 1)(x + 5) = 0$

$x^2 - 16 = 0$ → $(x - 4)(x + 4) = 0$

$12x^2 - x - 1 = 0$ → $(3x - 1)(4x + 1) = 0$

Example: **Building codes require that any storage shed be limited to a maximum "foot print" or area of 160 square feet. If you want to build a shed that is 12 feet longer than it is wide, what are the dimensions of the concrete base that must be constructed?**

Solution **Let x = the width.**

How do we represent the length? **Length = x + 12.**

Write a formula for the area:

Area = Length x Width = (x + 12) x = 160

Solve (x + 12) x = 160 → $x^2 + 12x - 160 = 0$

Factor this quadratic: (x   )(x   ) = 0

(x + 20)(x - 8) = 0

So x = -20 or x = 8

Which value do we choose for the width?

What are the dimensions? **Width = 8,**
then Length = width + 12 = 20

SQUARE ROOT PROPERTY

Let k be a nonnegative number. Then the solutions to the equation

$$x^2 = k$$

are given by $x = \pm\sqrt{k}$.

Example: $x^2 = 9 \rightarrow x = \pm\sqrt{9} = \pm 3$

$x^2 - 21 = 0 \rightarrow x^2 = 21$

$\rightarrow x = \pm\sqrt{21} \approx \pm 4.58$

Can we solve the quadratic $2x^2 + 32 = 0$?

Example:

A rescue helicopter hovers 68 feet above a jet ski in distress and drops a life raft. The height in feet of the raft above the water is given by $h(t) = -16t^2 + 68$.

Determine how long it will take for the raft to hit the water after being dropped from the helicopter.

Solution:

The raft will hit the water when its height is 0 feet above the water. Solve $h(t) = -16t^2 + 68 = 0$.

$$h(t) = -16t^2 + 68 = 0 \rightarrow -16t^2 = -68$$

$$\rightarrow t^2 = \frac{68}{16}$$

$$\rightarrow t = \pm\sqrt{\frac{68}{16}} \approx \pm 2.1 \text{ sec}$$

The life raft will hit about 2.1 seconds after it is dropped.

Ignore the -2.1. WHY?

QUADRATIC FORMULA

The solutions to the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example: **Solve $3x^2 - 5x + 2 = 0$ using the quadratic formula.**

a = 3, b = -5, c = 2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(2)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{25 - 24}}{6} = \frac{5 \pm 1}{6} = \begin{cases} 6/6 = 1 \\ 4/6 = 2/3 \end{cases}$$

x = 1 or x = 2 / 3

Example

Solve the equation $2x^2 - 5x - 9 = 0.$

Let *a* = 2, *b* = –5, and *c* = –9.

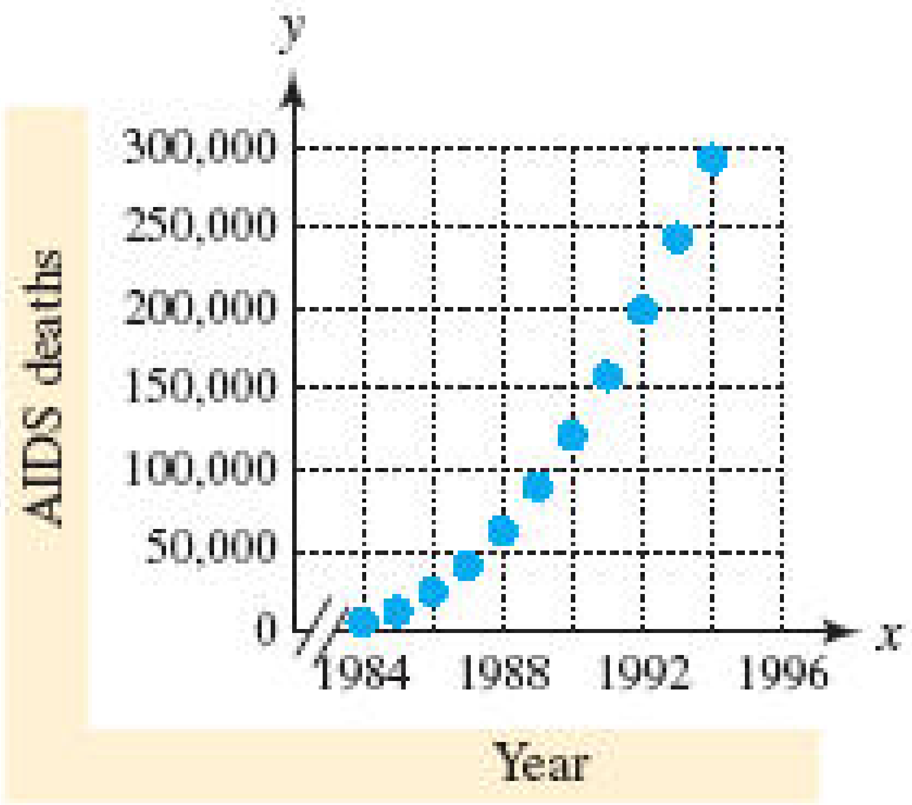
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-9)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{97}}{4}$$

Example: **The scatter plot below shows cumulative numbers of AIDS deaths in the US from 1984 to 1994. The plot was obtained by evaluating the quadratic** **$D(x) = 2375x^2 + 5134x + 5020$.** **(D(x) was found by quadratic regression on collected data.)**

In this model x = 0 corresponds to 1984 and x = 10 to 1994.



Estimate the year deaths reached 400,000.

Solve **$D(x) = 2375x^2 + 5134x + 5020 = 400{,}000$**

$$2375x^2 + 5134x + 5020 - 400{,}000 = 0$$

$$2375x^2 + 5134x - 394980 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(5134) \pm \sqrt{(5134)^2 - 4(2375)(-394980)}}{2(2375)}$$

$$= \frac{-(5134) \pm \sqrt{26357956 + 3752310000}}{2(2375)}$$

$$= \frac{-(5134) \pm \sqrt{3778667956}}{2(2375)} = \frac{-(5134) \pm 61470.87079}{2(2375)}$$

$$» \begin{cases} 11.86 \\ -14.02 \end{cases} » \begin{cases} 12 \\ -14 \end{cases}$$

→

Year when deaths reached 400,000 is 1984 – 14 = 1970 or 1984 + 12 = 1996.

Which do we “choose” here? WHY?

QUADRATIC EQUATIONS AND THE DISCRIMINANT

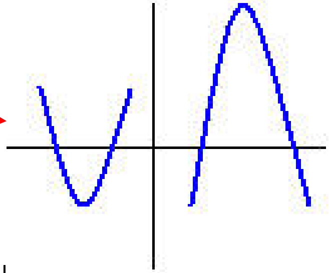
To determine the number of real solutions to $ax^2 + bx + c = 0$, with $a \neq 0$, evaluate the discriminant $b^2 - 4ac$.

1. If $b^2 - 4ac > 0$, there are two real solutions.
2. If $b^2 - 4ac = 0$, there is one real solution.
3. If $b^2 - 4ac < 0$, there are no real solutions.

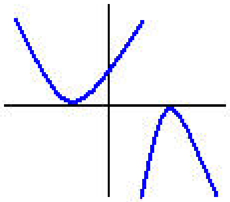
   Solutions will be complex numbers.

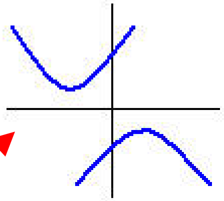
What does this imply about the graph of the parabola $y = ax^2 + bx + c$?

Two real solutions ➔ 2 x-intercepts



One real solution ➔ 1 x-intercept





No real solutions ➔ NO x-intercepts

Example: Use the discriminant to determine the number of solutions to the quadratic equation

$$-3x^2 - 6x + 15 = 0.$$

Compute $b^2 - 4ac$

$$b^2 - 4ac = (-6)^2 - 4(-3)(15) = 216$$

Since the discriminant is positive the equation has two real solutions.

What shape is the graph of the parabola $y = -3x^2 - 6x + 15$?

Modeling Projectile Motion

Example: The following table shows the height of a toy rocket launched in the air.

Height of a toy rocket			
t (sec)	0	1	2
s(*t*) feet	12	36	28

a) Use $s(t) = -16t^2 + v_o t + h_o$ to model the data.

b) After how many seconds did the toy rocket strike the ground?

Solution:

a) To determine the coefficients in the quadratic equation

$$s(t) = -16t^2 + v_o t + h_o$$

we use the data in the table

t	0	1	2
$s(t)$	12	36	28

Set t = 0 and s(t) = 12 ➔ $12 = h_0$

Now we have $s(t) = -16t^2 + v_0 t + 12$

Next set t = 1 and s(t) = 36 ➔ $36 = -16 + v_0 + 12$

Solve for v_0; we get $v_0 = 40$

So $s(t) = -16t^2 + 40t + 12$

Thus $s(t) = -16t^2 + 40t + 12$ models the height of the toy rocket.

Solution continued:

b) The rocket strikes the ground when

$s(t) = 0$, or when **$-16t^2 + 40t + 12 = 0$**.

Let's simplify before applying the quadratic formula.

$$-16t^2 + 40t + 12 = 0$$

$$4t^2 - 10t - 3 = 0$$

Using the quadratic formula for a = 4, b = – 10 and c = – 3 we find that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{100 + 48}}{8} = \frac{10 \pm \sqrt{148}}{8} \approx \frac{10 \pm 12.16}{8}$$

So to the nearest tenth **$x \approx 2.8$ or $x \approx -0.3$**

Only the positive solution is possible, so the toy rocket reaches the ground after approximately **2.8** seconds.

Building boxes by cutting squares from each corner of a rectangular piece of material.

Go to MATLAB; execute buildabox.

Investigate a more general example.

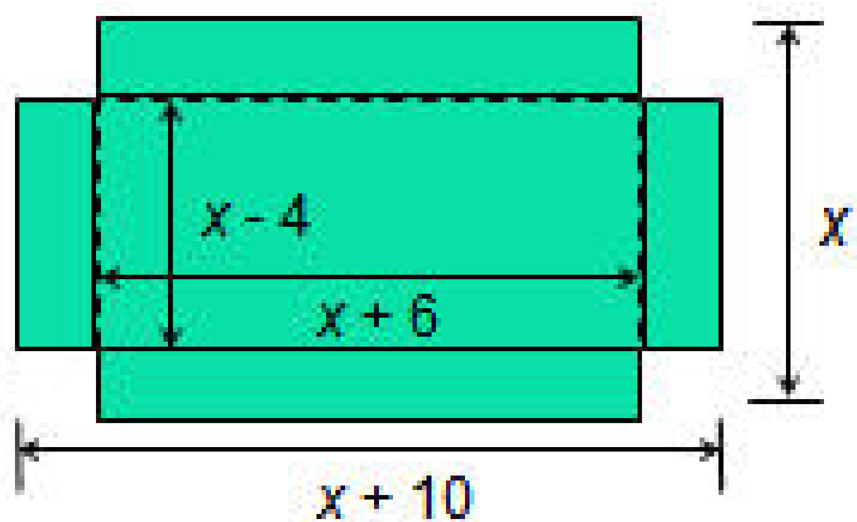
Example:

A box is being constructed by cutting 2 inch squares from the corners of a rectangular sheet of metal that is 10 inches longer than it is wide. If the box has a volume of 238 cubic inches, find the dimensions of the metal sheet.

Solution

Step 1: Let x be the width and $x + 10$ be the length.

Step 2: Draw a picture.



Solution continued

Step 3: Form an equation

Since the height times the width times the length must equal the volume, or 238 cubic inches, the following formula can be written

$$2(x-4)(x+6)=238 \text{ or}$$

$$(x-4)(x+6)=119$$

Write the quadratic equation in the form $ax^2 + bx + c = 0$ and factor.

$$x^2+2x-24=119$$

$$x^2+2x-143=0$$

$$(x+13)(x-11)=0$$

$$x=-13 \text{ or } x=11$$

The dimensions can not be negative, so the width is 11 inches and the length is 10 inches more, or 21 inches.

Solution continued

Step 4: After the 2 square inch pieces are cut out, the dimensions of the bottom of the box are 11 – 4 = 7 inches by 21 – 4 = 17 inches. The volume of the box is then V = 2(7)(17) = 238, which checks.