2.3

Linear Equations

This section focuses on symbolic, graphical and numerical techniques for solving equations.

- **Understand basic terminology related to equations**
- **Solve linear equations symbolically**
- **Solve linear equations graphically and numerically**
- **Understand the intermediate value property**
- **Solve problems involving percentages**
- **Apply problem-solving strategies**

PEARSON
Addison Wesley

Terminology:

An equation is a statement that two mathematical expressions are equal. (There is always an = sign involved.)

To solve an equation means to find all values for the variable that make the equation a true statement. Such values are called solutions. The set of all solutions is called the solution set.

The solution set of an equation can be empty (no solutions), contain exactly one value (unique solution), contain several values (finitely many solutions), or contain an infinite number of values.

Linear equations easier to solve than nonlinear equations.

Review:

Identify the equations as linear or nonlinear.

$$2(4 - 3x) + x = 3(x - 2)$$

$$x^2 + 2x - 8 = 0$$

$$50 - 12x = 20x - 8$$

$$2x - \sqrt{x} = 6$$

$$x(1 - x) + 3x + 2 = 1$$

Types of Equations in One Variable

- <u>Contradiction</u> – An equation for which there is no solution.

  Example: $2x + 3 = 5 + 4x - 2x$
  - Simplifies to $2x + 3 = 2x + 5$
  - Simplifies to $3 = 5$
  - FALSE statement – there are no values of x for which $3 = 5$. The equation has <u>NO SOLUTION</u>. <u>(Empty solution set.)</u>

- <u>Identity</u> – An equation for which every meaningful value of the variable is a solution.

  Example: $2x + 3 = 3 + 4x - 2x$
  - Simplifies to $2x + 3 = 2x + 3$
  - Simplifies to $3 = 3$
  - TRUE statement – no matter the value of x, the statement $3 = 3$ is true. The solution is ALL REAL NUMBERS. <u>(Infinitely many solutions.)</u>

Types of Equations in One Variable

Conditional Equation – An equation that is satisfied by *some*, but not all, values of the variable.

Example 1: $2x + 3 = 5 + 4x$

- **Simplifies to** $2x - 4x = 5 - 3$
- **Simplifies to** $-2x = 2$
- **Solution of the equation is:** $x = -1$ (Unique solution.)

Example 2: $x^2 = 1$

- **Solutions of the equation are:** $x = -1$, $x = 1$.
  (Nonlinear equation with exactly two solutions.)

Linear Equations in One Variable

A linear equation in one variable is an equation that can be rearranged to have the form $ax + b = 0$ where a and b are real numbers with $a \neq 0$.

Examples of linear equations in one variable:

$5x + 4 = 2 + 3x$ simplifies to $2x + 2 = 0$

$-1(x - 3) + 4(2x + 1) = 5$

simplifies to $7x + 2 = 0$

Examples of nonlinear equations in one variable.

$x^2 + 3x = 1$

$$\frac{1}{x-1} + x = 0$$

Example of Solving a Linear Equation Symbolically

Solve $-1(x - 3) + 4(2x + 1) = 5$ for x

Expand ➔ $-1x + 3 + 8x + 4 = 5$

Collect like terms ➔ $7x + 7 = 5$

Rearrange ➔ $7x = 5 - 7$

Simplify ➔ $7x = -2$

Solve for x ➔ $x = -2/7$ **← Exact Solution**

Linear Equations can always be solved symbolically.

Linear Equations have only THREE possible solution sets:

Empty (no solution)

Unique solution (just one value)

Infinitely many values

Example of Solving a Linear Equation Involving Fractions Symbolically

Solve $\frac{x-1}{3}+5=\frac{1}{4}$

Solution Process:

$$12\left(\frac{x-1}{3}+5\right)=12\left(\frac{1}{4}\right)$$

Multiply $\Rightarrow 4(x-1)+60=3$

Expand $\Rightarrow 4x-4+60=3$

Collect like terms $\Rightarrow 4x+56=3$

Rearrange $\Rightarrow 4x=3-56$

Simplify $\Rightarrow 4x=-53$

Solve for x $\Rightarrow x=\frac{-53}{4}=-13.25$

- When solving a linear equation involving fractions, it is often helpful to multiply both sides by the least common denominator of all of the denominators in the equation.
- The least common denominator of 3 and 4 is 12.

🡐 Exact Solution

Solving a Linear Equation Graphically (Intersection of Graphs Method)

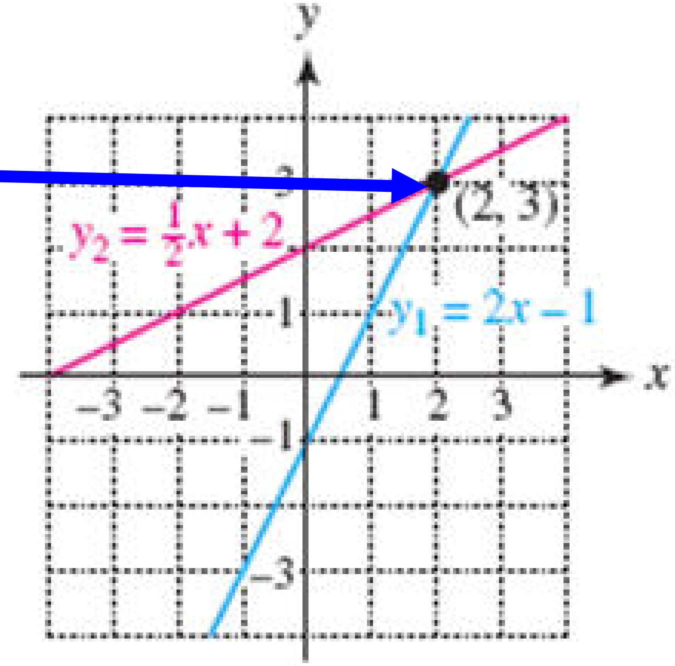
Example: Solve $2x - 1 = \frac{1}{2}x + 2$

Step 1. Let y_1 equal the left side and y_2 equal the right side.

$y_1 = 2x - 1$ $y_2 = \frac{1}{2}x + 2$

Step 2. Graph both expressions on the same set of axes.

Step 3. Locate any intersection points. The x-coordinates of these points are solutions.



Warning: To do this on a calculator or computer you may need to adjust the sketching window.

Another graphical procedure.

Many calculators and some computer programs have a "trace curve" feature.

You graph a function and then can have a dot or cross hair move along the curve displaying (approximate) coordinates of points.

Example:

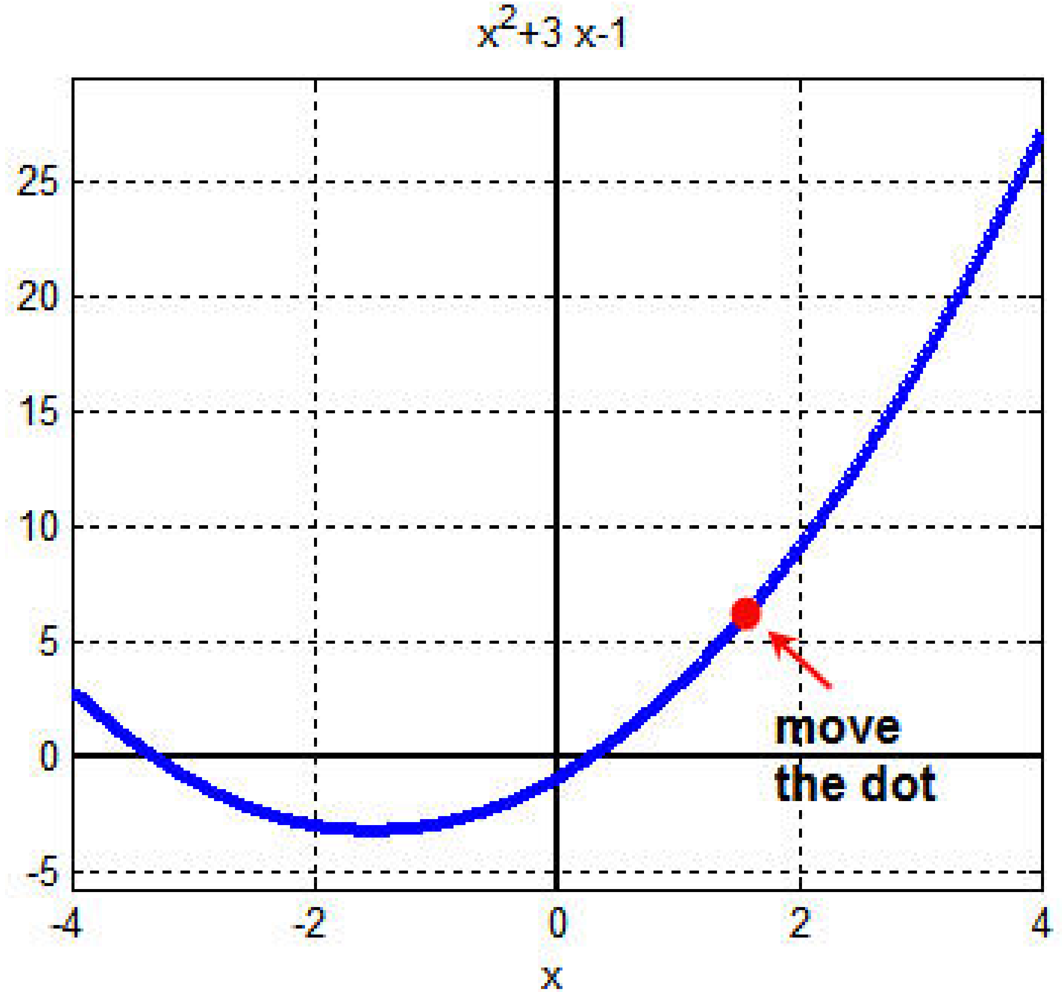
To approximate solutions of nonlinear equation $x^2 + 3x = 1$ we rewrite the expression as $x^2 + 3x - 1 = 0$, graph function $f(x) = x^2 + 3x – 1$, and then estimate its x-intercepts.

Example:

To approximate solutions of nonlinear equation $x^2 + 3x = 1$ we rewrite the expression as $x^2 + 3x - 1 = 0$, graph function $f(x) = x^2 + 3x - 1$, and then estimate its x-intercepts.

As you move the dot its x- and y-coordinates are displayed.

The values at a position are only approximate.



Software Demonstrations:

Estimate the intercepts of the graph of $y = x^2 + 3x - 1$ over interval [-4, 4].

A Java Applet from National Library of Virtual Manipulatives.

A Flash program.

Each program has different input rules and different features.

How do you check the accuracy of the approximations?

Numerical solution of equations.

If the equation is linear the symbolic method (the use of algebra) is preferred.

If the equation is nonlinear then the graphical method can be used to get an estimate of solutions, but those values will probably be accurate to just a few decimal places.

A numerical approach usually consists of determining a table of values. The approach taken in the text is as follows:

(a) Let y_1 equal the left side and y_2 equal the right side.

(b) Choose values of x and compute corresponding values of y_1 and y_2.

(c) "Hope" you do this enough so that y_1 and y_2 are close so you can estimate a solution value x.

Numerical solution of equations. (continued)

With a few ideas, the computation of function values, and the computation of midpoints for intervals we can get quite accurate solutions to many nonlinear equations.

The ideas we need.

1. **A continuous function is one whose graph can be sketched (or traced) without picking up your pencil.**

2. **The Intermediate Value Property:**

Let (x_1, y_1) and (x_2, y_2) with $y_1 \neq y_2$ and $x_1 < x_2$ be two points on the graph of a continuous function f. Then, on the interval $x_1 \leq x \leq x_2$, f assumes every value between y_1 and y_2 at least once

Examples involving the intermediate value property.

If a continuous curve connects points (3, -4) and (7,2) then it must cross the x-axis at least once on the interval [3, 7].

Reason: It was below the x-axis at x = 3 and above at x = 7 and we must be able to connect the points without "lifting our pencil".

If a car is going 20 mph and a short time later it is going 40 mph, then at some time it was going 35 mph.

Reason: It is reasonable to assume that the car is in continuous motion and the intermediate value property applies.

Problem-Solving – Modeling with Linear Equations

- **STEP 1:** Read the problem and make sure you understand it. Assign a variable to what you are being asked. If necessary, write other quantities in terms of the variable.

- **STEP 2:** Write an equation that relates the quantities described in the problem. You may need to sketch a diagram and refer to known formulas.

- **STEP 3:** Solve the equation and determine the solution.

- **STEP 4:** Look back and check your solution. Does it seem reasonable?

Modeling with Linear Equations – Solving an Application Involving Motion

- **In 2 hours an athlete travels 18.5 miles by running at 11 miles per hour and then by running at 9 miles per hour. How long did the athlete run at each speed?**

- **STEP 1:** We are asked to find the time spent running at each speed. If we let x represent the time in hours running at 11 miles per hour, then $2 - x$ represents the time spent running at 9 miles per hour.

  x: Time spent running at 11 miles per hour

  $2 - x$: Time spent running at 9 miles per hour

- **STEP 2:** Distance d equals rate r times time t: that is, $d = rt$. In this example we have two rates (speeds) and two times. The total distance must sum to 18.5 miles.

$$d = r_1t_1 + r_2t_2$$

$$18.5 = 11x + 9(2 - x)$$

- **STEP 3:** Solving $18.5 = 11x + 9(2 - x)$ symbolically

$$18.5 = 11x + 18 - 9x$$
$$18.5 - 18 = 2x$$
$$0.5 = 2x$$
$$x = 0.5/2$$
$$x = 0.25$$

The athlete runs .25 hours (15 minutes) at 11 miles per hour and 1.75 hours (1 hours and 45 minutes) at 9 miles per hour.

- **STEP 4:** We can check the solution as follows.

$$11(.25) + 9(1.75) = 18.5 \quad \text{(It checks.)}$$

This sounds reasonable. The average speed was 9.25 mi/hr, that is 18.5 miles/2 hours. Thus the runner would have to run longer at 9 miles per hour than at 11 miles per hour, since 9.25 is closer to 9 than 11.

Modeling with Linear Equations – Mixing Acid in Chemistry

- **Pure water is being added to a 25% solution of 120 milliliters of hydrochloric acid. How much water should be added to reduce it to a 15% mixture?**
- **STEP 1:** We need the amount of water to be added to 120 milliliters of 25% acid to make a 15% solution. Let this amount of water be equal to x.

  x: Amount of pure water to be added

  $x + 120$: Final volume of 15% solution
- **STEP 2:** Construct an equation.

25%		0%		15%
120 ml	+	x ml	=	120 + x ml

The total amount of acid in the solution must balance.

$$.15(x + 120) = .25(120)$$

- **STEP 3:** **Solving $.15(x + 120) = .25(120)$ symbolically**

$$.15x + 18 = 30$$
$$.15x = 12$$
$$x = 12/.15$$
$$x = 80 \text{ milliliters}$$

- **STEP 4:** **This sounds reasonable. If we added 120 milliliters of water, we would have diluted the acid to half its concentration, which would be 12.5%. It follows that we should not add much as 120 milliliters since we want a 15% solution.**

Modeling with Linear Equations – Population Density

In 1980 the population density of the United States was 64 people per square mile and in 1990 it was 70 people per square mile. Use a linear function to estimate when the U.S. population density reached 72.4 people per square mile.

Step 1. Estimate the year in which the population density reached 72.4 per square mile.

Let x = be the time in years and y = density of the population.

Step 2. When x = 1980, y = 64 and when x = 1990, y = 70, so we have two data points.

Create a linear equation: Find the slope.

$$\text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{70 - 64}{1990 - 1980} = \frac{6}{10} = 0.6$$

Equation: $y - 64 = 0.6(x - 1980)$ ← Point slope form.

Step 3: Solve $y - 64 = 0.6(x - 1980)$ symbolically when y = 72.4.

$72.4 - 64 = 0.6(x - 1980)$ ← Set y = 72.4

$8.4 = 0.6x - 1188$ ← Expand and simplify

$8.4 + 1188 = 0.6x$ ← Collect like terms

$1196.4 = 0.6x$ ← Simplify

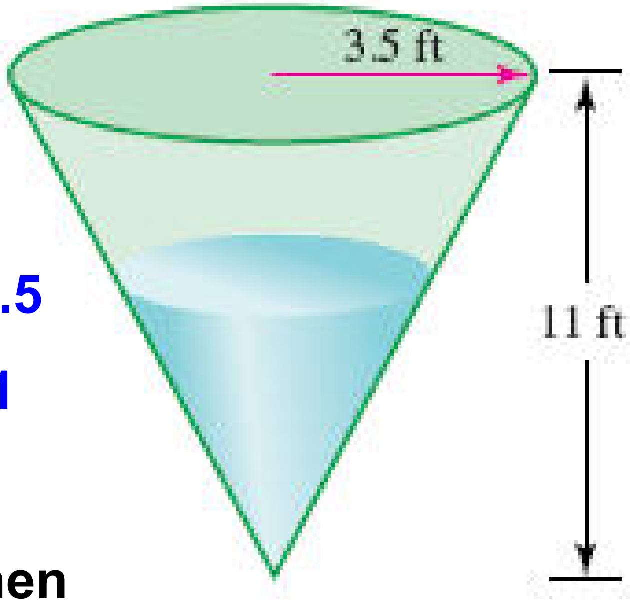
$\frac{1196.4}{0.6} = x$ ← Solve for x.

$x = 1994$

Step 4: The change over 10 years was 6 people per square mile. Since the slope is 0.6 that implies that there will be an increase in density of 0.6 people per year; 4 times 0.6 = 2.4 is the increase from 1990 to 1994. So x = 1994 seems reasonable.

Is this process interpolation or extrapolation?

Modeling with Linear Equations – A Conical Tank of Water



A water tank has the shape of an inverted cone as shown in the figure.

The volume of a cone is

$$V = \frac{1}{3}\pi r^2 h$$

Radius = 3.5

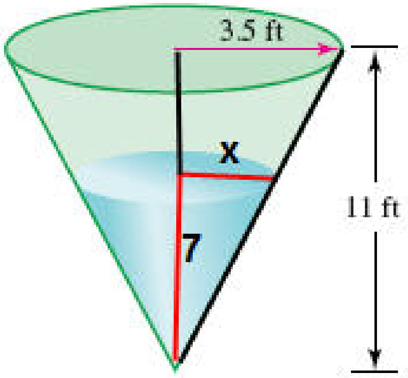
Height = 11

Find the volume of the water in the tank when the depth is 7 feet.

Step 1: To compute the volume we need the radius when the height is 7 feet.

Let x = radius when h = 7.

Step 2: To find x we will use similar triangles.



$$\frac{x}{3.5} = \frac{7}{11}$$

Step 3: Solve for x in $$\frac{x}{3.5} = \frac{7}{11}$$

$$x = \frac{3.5(7)}{11}$$ ← Cross multiply.

$$x = \frac{24.5}{11}$$ ← Multiply

$x = 2.23$ (approximately) ← Divide

When the depth of the water is 7 feet, the radius is (about) 2.23 feet.

So the volume is $V = \frac{1}{3}\pi(2.23)^2\,7 \approx 36.4$ cubic feet

Step 4: The volume of a full tank is (about) 141.11 cubic feet and the volume with 7 feet of water should be quite a bit smaller based on the shape of the cone, so the result seems reasonable.

Modeling with Linear Equations – Housing Costs

The following table shows the average costs of new one-family homes in a certain neighborhood for three selected years.

Year	1999	2000	2001
Price in $1000	173	179	186

If a house was worth $200 thousand in 2001, estimate its worth in 1999.

(a) Find the percent of increase in sale price from 1999 to 2000.

(b) Find the percent of increase in sale price from 2000 to 2001.

(c) Use the results in parts (a) and (b) to estimate the worth of a house in1999 given that is was worth $200,000 in 2001.

Year	1999	2000	2001
Price in $1000	173	179	186

(a) Find the percent of increase in sale price from 1999 to 2000.

$$\frac{179-173}{173}\times 100 = 3.46\%$$

(b) Find the percent of increase in sale price from 2000 to 2001.

$$\frac{186-179}{179}\times 100 = 3.91\%$$

(c) Use the results in parts (a) and (b) to estimate the worth of a house in1999 given that is was worth $200,000 in 2001.

The price will decrease by 3.91% from 2001 to 2000 and then by 3.46% from 2000 to 1999.

(1 - .0391)200 =192.18 ← price in 2000

(1 - .0346)192.18 =185.5 ← price in 1999