1.4

Types of Functions and Their Rates of Change

- Identify and use constant and linear functions
- Interpret slope as a rate of change
- Identify and use nonlinear functions
- Recognize linear and nonlinear data
- Use and interpret average rate of change
- Calculate the difference quotient

INTRODUCTION

A central theme of applied mathematics is describing real-world phenomena with functions. Because applications involving real-world data are diverse, mathematicians have created a wide assortment of functions, In fact, professional mathematicians design new functions every day for use in business, education, and government. Mathematics is not static – it is dynamic; it grows and changes. It requires both ingenuity and creativity to analyze data and make predictions about the future.

In this section we introduce basic types of data and functions that provide models for the data.

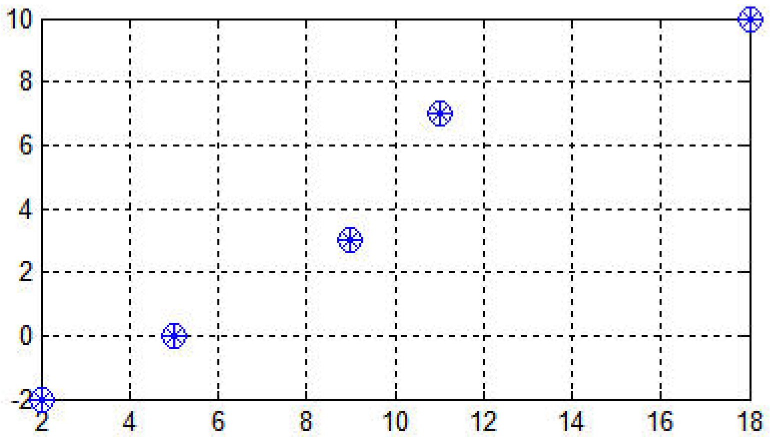
Two particular types of functions.

1. **Discrete Function:** A function whose **domain** is a set of discrete values.

Example:

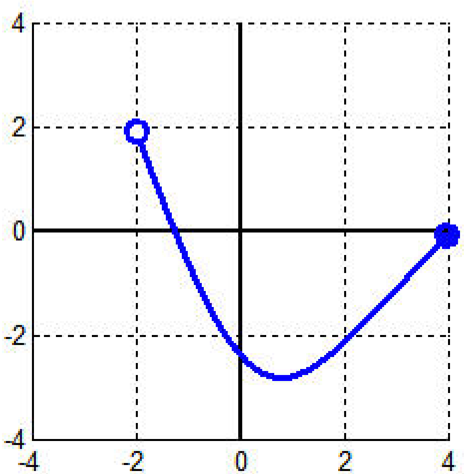
x	2	5	9	11	18
y	-2	0	3	7	10

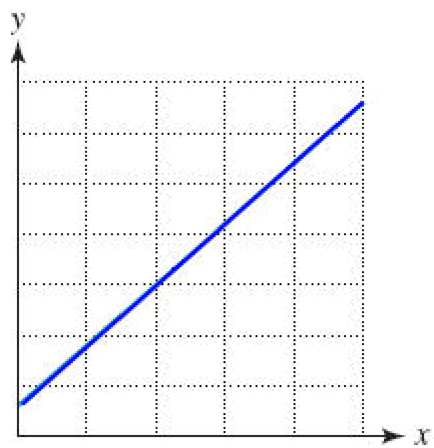
The graph of a discrete function is a **scatterplot**.

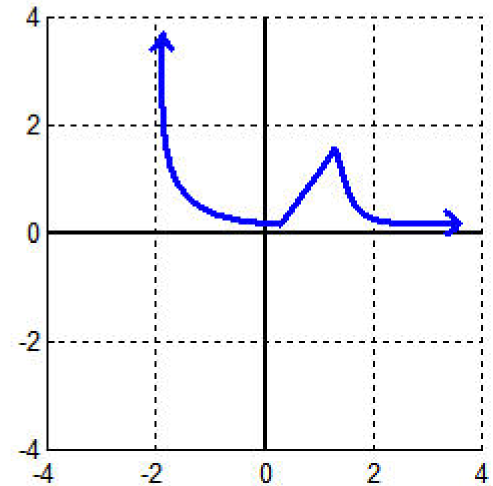


2. <u>Continuous Function:</u> A function whose **graph can be sketched (or traced) <u>without</u> picking up your pencil.**

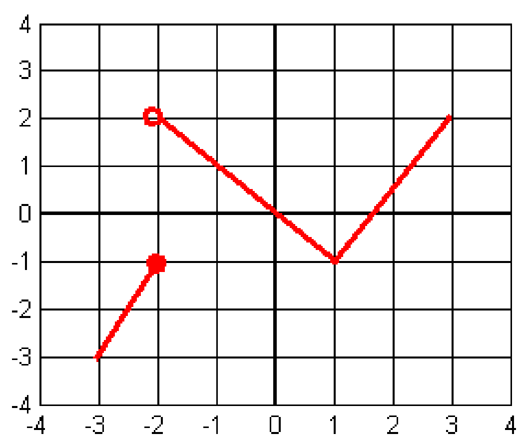
<u>Examples:</u>

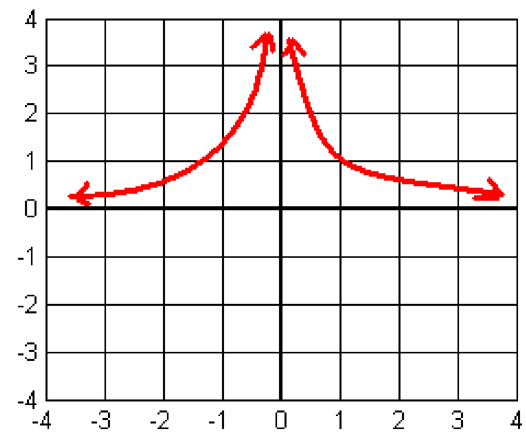






Functions which are not continuous.





Constant Function

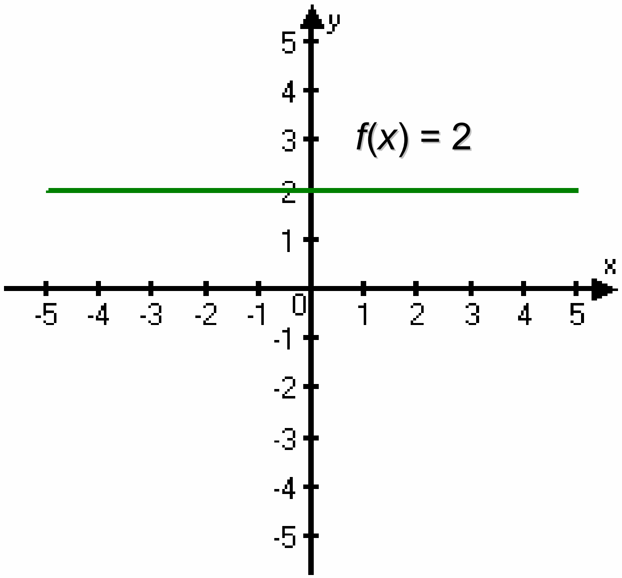
- **A function *f* represented by *f*(*x*) = *b*, where *b* is a constant (fixed number), is a constant function.**

Examples:

$$f(x) = 2$$

$$f(x) = \frac{-1}{2}$$

$$f(x) = \sqrt{2}$$

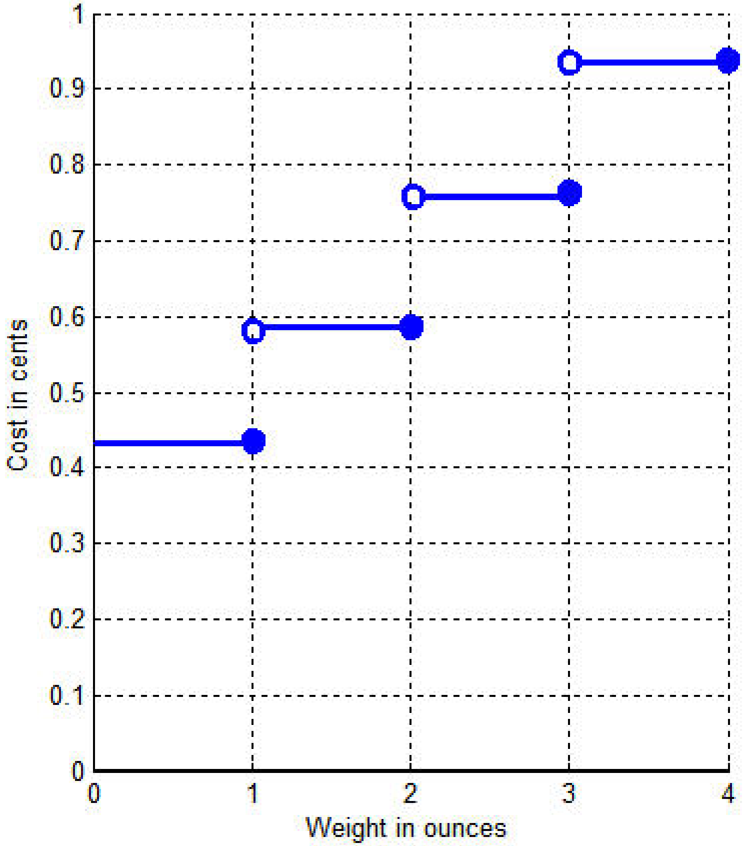


The graph of a constant function is a horizontal line.

A model using constant functions

Retail postage rates for letters.

Weight Not Over (Ounces)	Cost of single piece
1	$0.42
2	$0.59
3	$0.76
3.5	$0.93



Is this a continuous function?

Question: What was the postal rate for a letter in 1950?

Domestic Letter Rate: 3¢ per oz

Linear Function

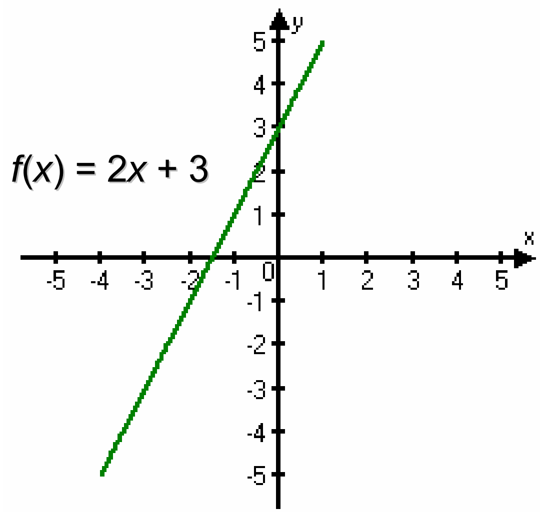
- **A function *f* represented by *f*(*x*) = *ax* + *b*, where *a* and *b* are constants, is a linear function**

Examples:

$f(x) = 2x + 3$ (Note: $a = 2$ and $b = 3$)

$f(x) = -5x - \frac{1}{2}$ $\left(\text{Note: } a = -5 \text{ and } b = -\frac{1}{2}\right)$

$f(x) = -6$ (Note: $a = 0$ and $b = -6$)



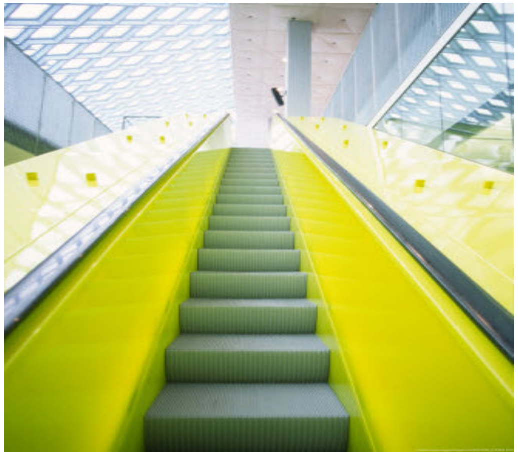
Note that a *f*(*x*) = -6 is both a linear function and a constant function.
A constant function is a special case of a linear function.

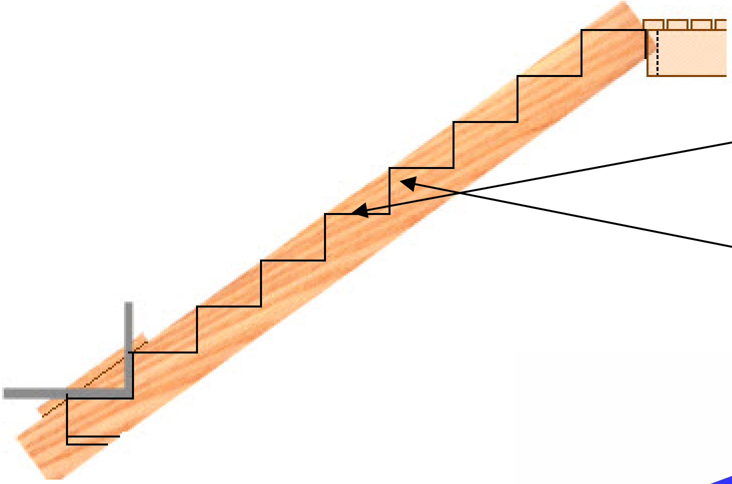
The graph of <u>any</u> linear function is a straight line.

<u>QUESTION:</u> Does every straight line represent a function?

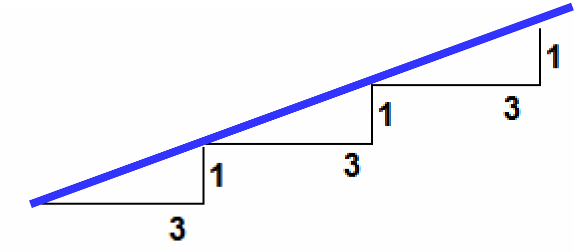
Constructing a Linear Function

A linear function is like constructing stairs or steps.





You move horizontally (the run), then you move vertically (the rise); now repeat using the same values.



Rate of Change of a Linear Function

Table of values for $f(x) = 2x + 3$.

x	y
–2	–1
–1	1
0	3
1	5
2	7
3	9

Note throughout the table, as x increases by 1 unit, y increases by 2 units.

In other words, the RATE OF CHANGE of y with respect to x is constantly 2 throughout the table.

Since the rate of change of y with respect to x is *constant*, the function is LINEAR.

Another name for rate of change of a linear function is SLOPE.

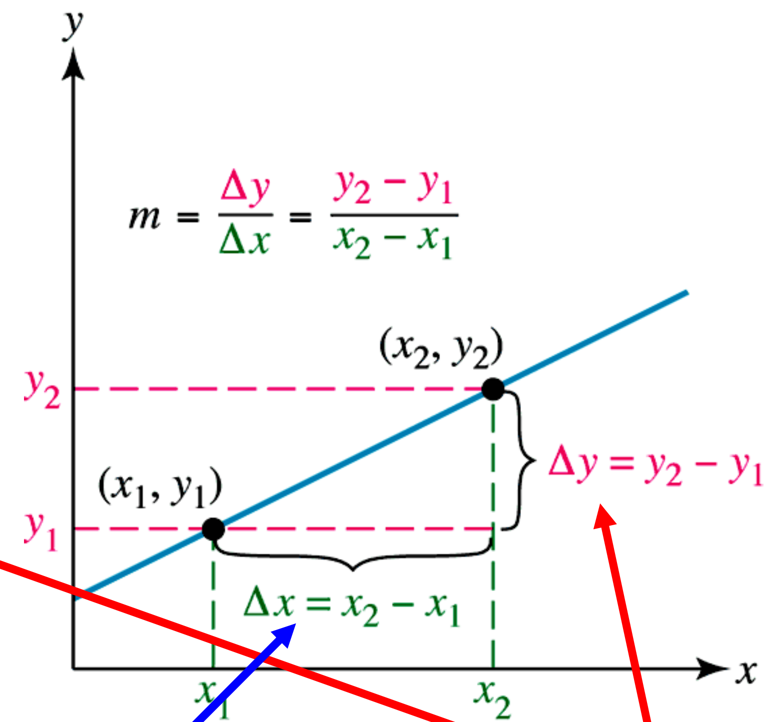
Here the run is 1 unit and the rise is 2 units.

Slope of Linear Function

The **slope** ***m*** of the line passing through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 \neq x_2$



Change in y

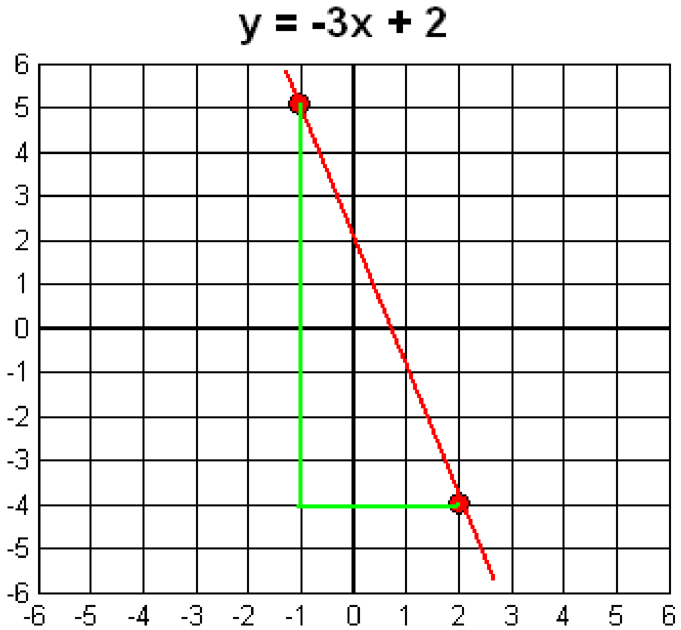
Change in x

Example of Calculation of Slope

Find the slope of the line passing through the points (–1, 5) and (2, -4).

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

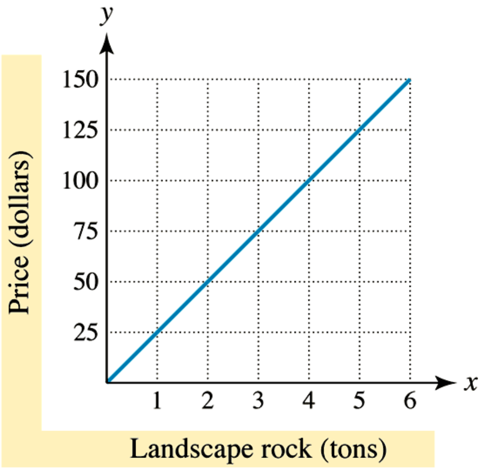
$$= \frac{-4 - 5}{2 - (-1)} = \frac{-9}{3} = -3$$



The slope being -3 means that for each unit x increases, the corresponding decrease in *y* is 3. The rate of change of *y* with respect to x is -3/1 or -3.

Another Example of a Linear Function

x (tons)	y (price in dollars)
1	25
2	50
3	75
4	100



Since for each 1 ton change the price changes by \$25, *y* is a linear function of *x*. Then the slope can be computed using any two points.

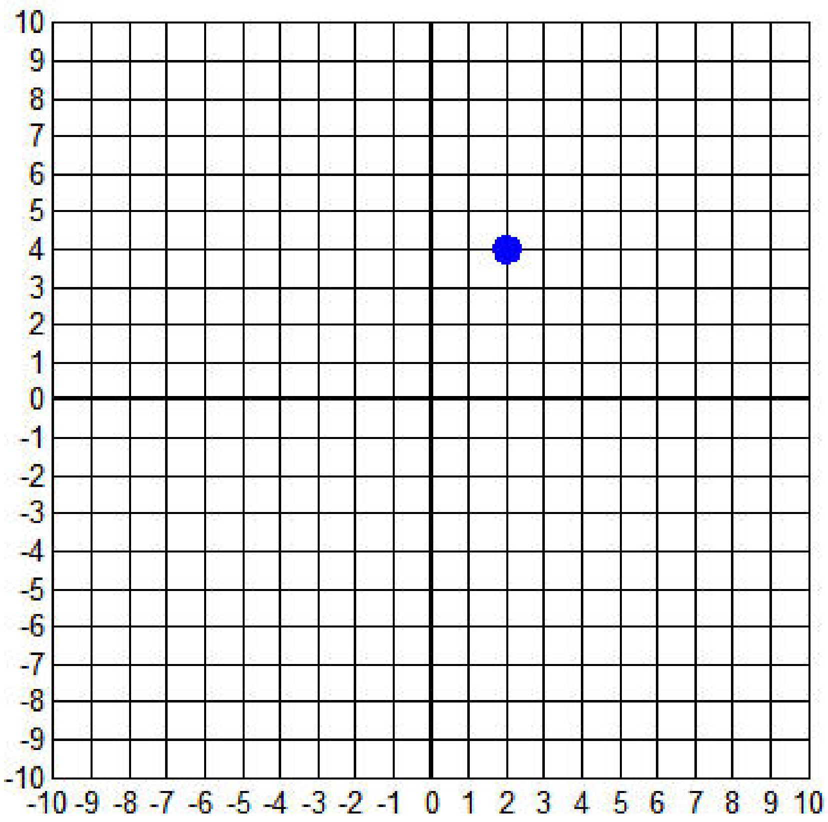
$$\frac{\Delta y}{\Delta x} = \frac{75-25}{3-1} = \frac{50}{2} = 25$$

$$\frac{\Delta y}{\Delta x} = \frac{50-25}{2-1} = 25$$

So the rate of change of price *y* with respect to tonnage *x* is 25 to 1.

Example using slope.

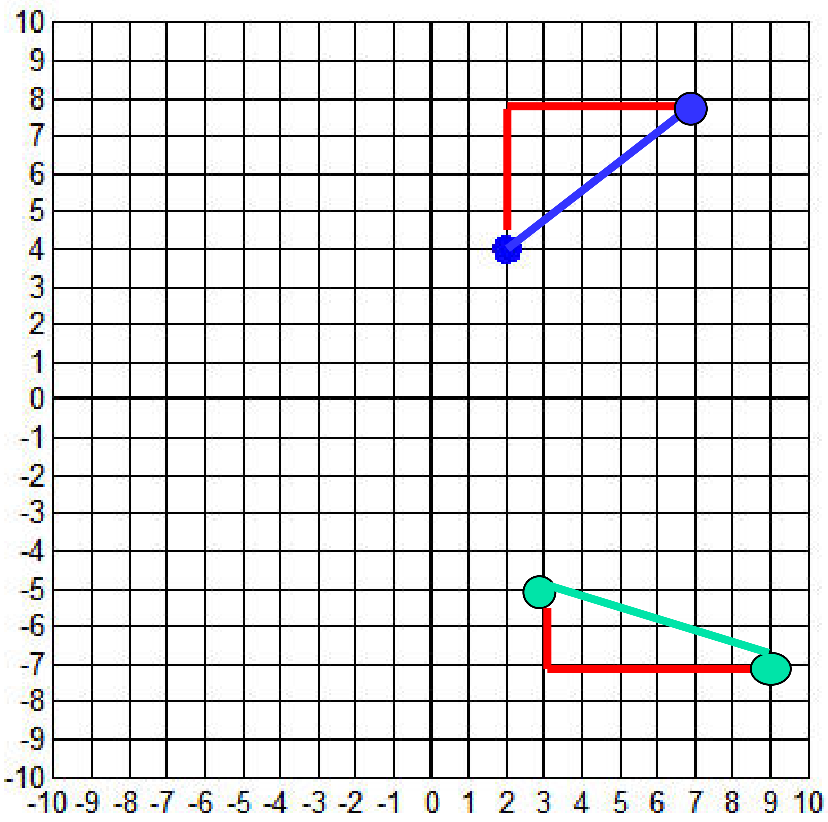
Suppose f is a continuous linear function that contains the point (2, 4) and has slope 4/5. Determine another point on the graph of f.



Example using slope.

Suppose f is a continuous linear function that contains the point (2, 4) and has slop 4/5. Determine another point on the graph of f.

$$\frac{\Delta y}{\Delta x} = \frac{4}{5}$$



Suppose the point was (3, -5) and the slope was -2/6.

$$\frac{\Delta y}{\Delta x} = \frac{-2}{6}$$

Identifying the slope in a linear equation.

A linear function *f* is represented by $f(x) = ax + b$.

The slope of the linear function is the value of a.

Identify the slope for each of the following lines.

- $f(x) = 7 - 3x$
- $y = 6$
- $x = 5$
- $g(x) = 8x+2$

Describe the graph of a linear function whose slope is negative.

Paper Experiment.

Partner with a person sitting next to you.

Each take out a sheet of paper. One of you will be the "folder" & the other the "recorder".

The "recorder" on their paper makes the following chart:

Fold number	Number of parts
0	
1	
2	
3	
4	

Directions will be given.

Can the data in the table be modeled by a linear function?

BB

Nonlinear functions.

If a function is not linear, then it is called nonlinear.

The graph of a nonlinear function is not a (straight) line.

We can infer that the $\dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\Delta y}{\Delta x}$ **for different pairs of points is not always the same value when the function is nonlinear.**

x = Fold number	y = Number of parts
0	1
1	2
2	4
3	8
4	16

Let's compute

$$\frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x}$$

for this data.

Many real-world phenomena are modeled by nonlinear functions. Some examples:

- **Spread of disease**
- **Monthly average temperatures in Philadelphia**
- **Yearly height of Kobe Bryant since birth**
- **Population of the US**
- **Daily price of a barrel of crude oil**
- **Dow-Jones average**
- **Monthly home heating bills**

Graphs of nonlinear functions are quite varied.

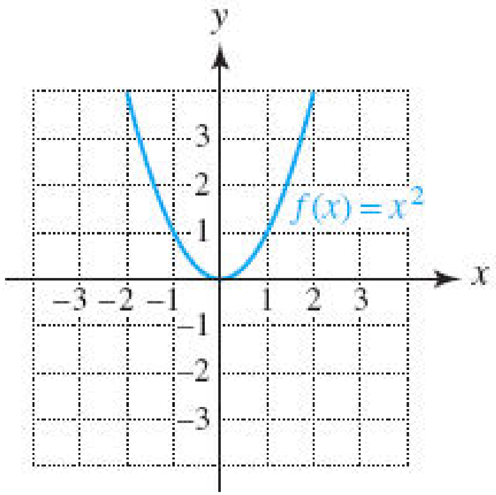


FIGURE 1.73 Square Function

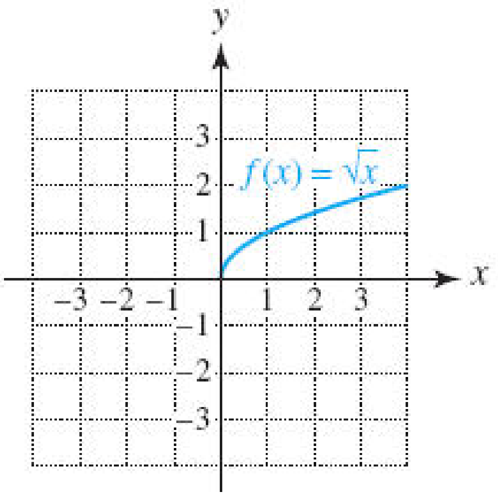


FIGURE 1.74 Square Root Function

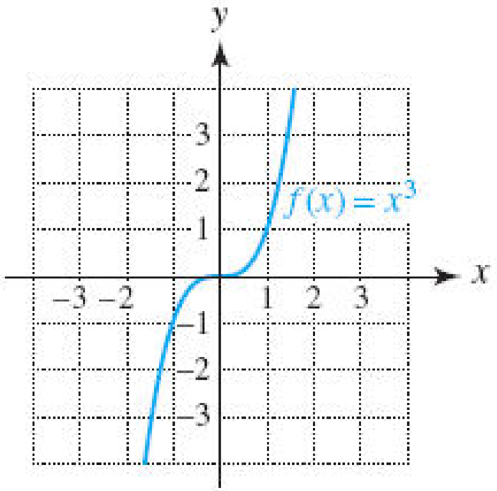


FIGURE 1.75 Cube Function

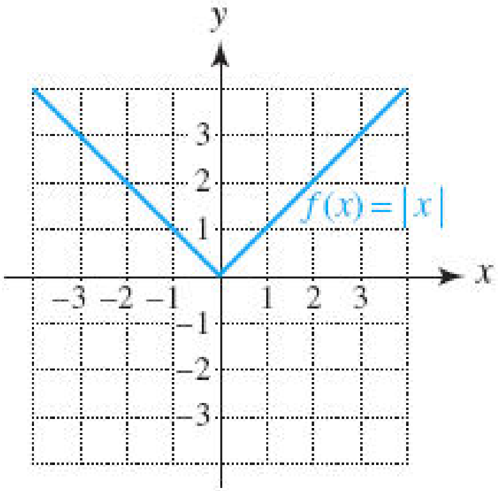


FIGURE 1.76 Absolute Value Function

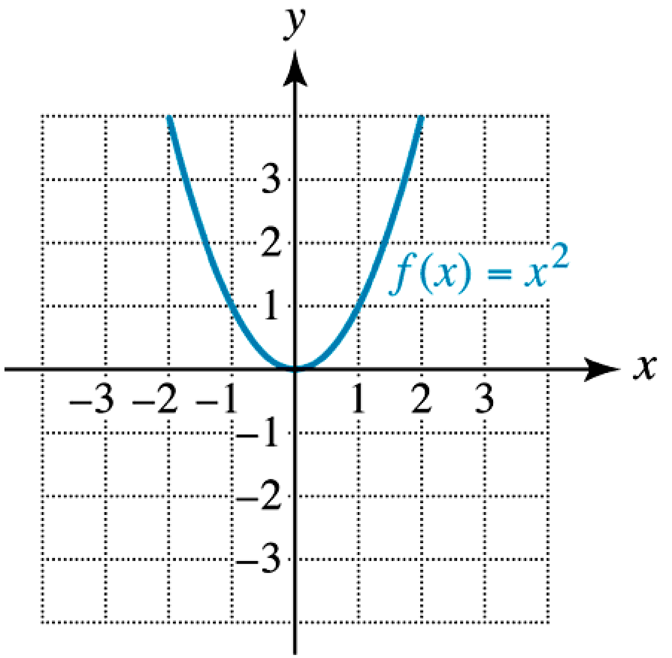
Example of a Nonlinear Function

Table of values for $f(x) = x^2$

x	y
0	0
1	1
2	4

$\Delta x = 1$, $\Delta y = 1$

$\Delta x = 1$, $\Delta y = 3$



Note that as x increases from 0 to 1, y increases by 1 unit; while as x increases from 1 to 2, y increases by 3 units.

This function does NOT have a CONSTANT RATE OF CHANGE of y with respect to x, so the function is NOT LINEAR.

Note that the graph is not a line.

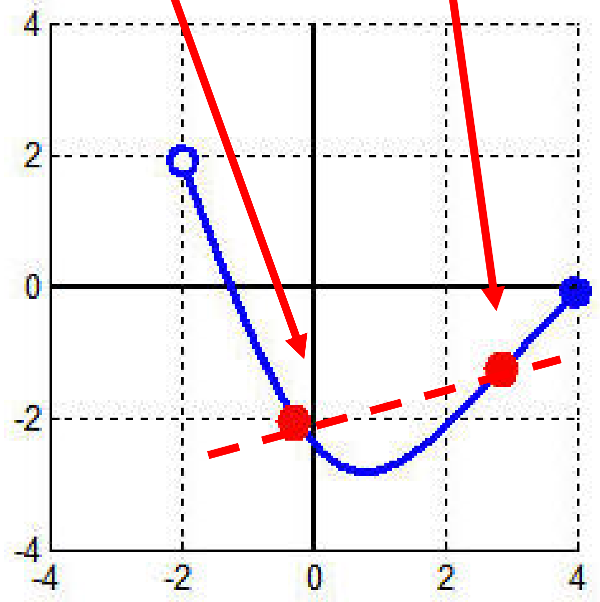
Average Rate of Change

Let (x_1, y_1) and (x_2, y_2) be distinct points on the graph of any function *f*. The average rate of change of *f* from x_1 to x_2 is $\dfrac{y_2 - y_1}{x_2 - x_1}$

Note that the average rate of change of f from x_1 to x_2 is the slope of the line passing through (x_1, y_1) and (x_2, y_2).

Question: What type of function has the average rate of change the same for every pair of distinct points?

To show the average rate of change between distinct points (x_1, y_1) and (x_2, y_2) on the graph of a function *f*, just draw the straight line connecting the points. Then compute its slope.



$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example of average rate of change.

On wet level pavement highway engineers sometimes model the braking distance in feet for a car traveling at x miles per hour by using the formula

$$f(x) = \frac{1}{9}x^2$$

Determine the average rate of change in breaking distance between 30 mph and 60 mph.

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Determine the average rate of change in breaking distance between 30 mph and 60 mph.

$$\frac{\Delta y}{\Delta x} = \frac{f(60) - f(30)}{60 - 30} = \frac{\frac{1}{9}(60)^2 - \frac{1}{9}(30)^2}{60 - 30}$$

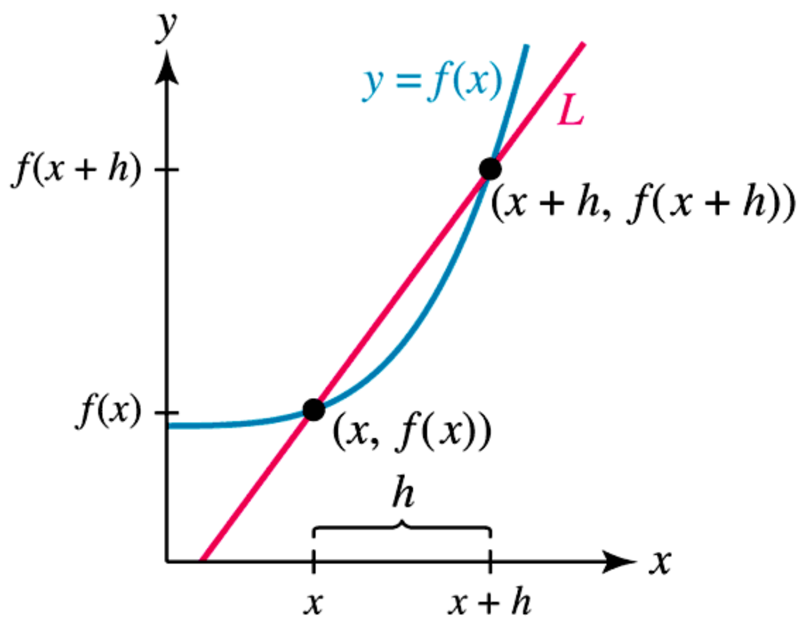
$$= \frac{400 - 100}{30} = \frac{300}{30} = 10$$

This means that the braking distance increases on average, by 10 feet for each 1-mile-per-hour increase in speed between 30 and 60 miles per hour.

The Difference Quotient

- **The difference quotient of a function *f* is an expression of the form $\frac{f(x+h)-f(x)}{h}$ where *h* is not 0.**

Note that a difference quotient is actually an average rate of change.



Calculating a difference quotient.

The distance d in feet that a racehorse travels after t seconds is given by $d(t) = 3t^2$. (Here take the domain to be [0, 10].)

Find the formula for the horse's average rate of change over the interval [t, t+h] where h is a positive value.

Compute the difference quotient $\dfrac{d(t+h)-d(t)}{t+h-t}$

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$$\frac{d(t+h)-d(t)}{t+h-t} = \frac{3(t+h)^2 - 3t^2}{h} = \frac{3\left(t^2 + 2ht + h^2\right) - 3t^2}{h}$$

$$= \frac{3t^2 + 6th + 3h^2 - 3t^2}{h} = \frac{6th + 3h^2}{h} = 6t + 3h$$

If t =2 sec. and h = 0.1 sec. what is the horse's average rate of change over the time interval [2, 2.1]?

12.3 ft/sec

Example of Calculating a Difference Quotient

- Let $f(x) = x^2 + 3x$. Find the difference quotient of f and simplify the result.

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$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 + 3(x+h) - (x^2+3x)}{h} =$$

$$\frac{(x^2+2xh+h^2)+3x+3h-x^2-3x}{h} = \frac{2xh+h^2+3h}{h} =$$

$$\frac{h(2x+h+3)}{h} = 2x+h+3$$