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PIECEWISE LOGLINEAR ESTIMATION OF EFFICIENT PRODUCTION SURFACES*

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Linear programming formulations for piecewise loglinear estimation of efficient production surfaces are derived from a set of basic properties postulated for the underlying production possibility sets. Unlike the piecewise linear model of Banker, Charnes, and Cooper (1984), this approach permits the identification of increasing marginal products, and estimation of the classical S-shaped production functions. Methods are also provided for estimating technical inefficiencies and other production characteristics, such as rates of substitution and transformation, marginal products, returns to scale and most productive scale sizes on the basis of observed production data. The results of a simulation study are reported to illustrate the usefulness of this estimation method in empirical applications when there are *a priori* reasons to expect increasing marginal products in some regions of the production function. A modified model is provided to extend this analysis to the situation of non-competing outputs addressed by the bi-extremal model of Banker, Charnes, Cooper, and Schinnar (1981).
(PRODUCTION FRONTIERS; DATA ENVELOPMENT ANALYSIS)

1. Introduction

Data envelopment analysis (DEA) provides a new nonparametric approach to the estimation of efficient production correspondences. Banker, Charnes, and Cooper (BCC) (1984) presented an axiomatic development of the estimation procedure. Postulating convexity and monotonicity for the production possibility set, they describe a linear programming formulation to estimate technical and scale inefficiencies, rates of substitution and transformation, marginal products and most productive scale sizes. The convexity postulate of BCC permits increasing, constant or decreasing returns to scale in different regions of the production function. However, this also requires the marginal products[1] to be nonincreasing, i.e., $(\partial/\partial x_i)(\partial y_r/\partial x_i) \leqslant 0$ for all y_r, x_i, at all points on the efficient production surface, where y_r, $r = 1, \ldots s$, are s outputs and x_i, $i = 1, \ldots m$, are m inputs. In fact, (ordinary) convexity requires that $(\partial/\partial x)(\partial y/\partial x) \leqslant 0$ at all points (yY_0, xX_0) on the efficient production surface, where x and y are scalars and $X_0 \equiv (x_{i0}, \ldots x_{m0})$ and $Y_0 \equiv (y_{i0}, \ldots y_{s0})$ are vectors representing specific input and output mixes.

This restriction in the BCC approach requiring nonincreasing marginal products may not be appropriate for production technologies of the type described in Figure 1, where the production function is nonconcave in some regions and the production possibility set is not convex. Efficient points like B and C will be rated as inefficient by the piecewise linear estimation method of BCC in this case. Existence of a fixed input and gains from increasing specialization with larger scale sizes are the usual economic

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[1]See Menger (1954) for an illuminating comparison of return to scale and rate of change of marginal product. Note that we do not assume in this paper that the production function is differentiable. However, we use partial derivatives for simplicity in exposition. See the discussion before definitions in 8(a)–8(d) for a rigorous definition of these notions in terms of subgradients.

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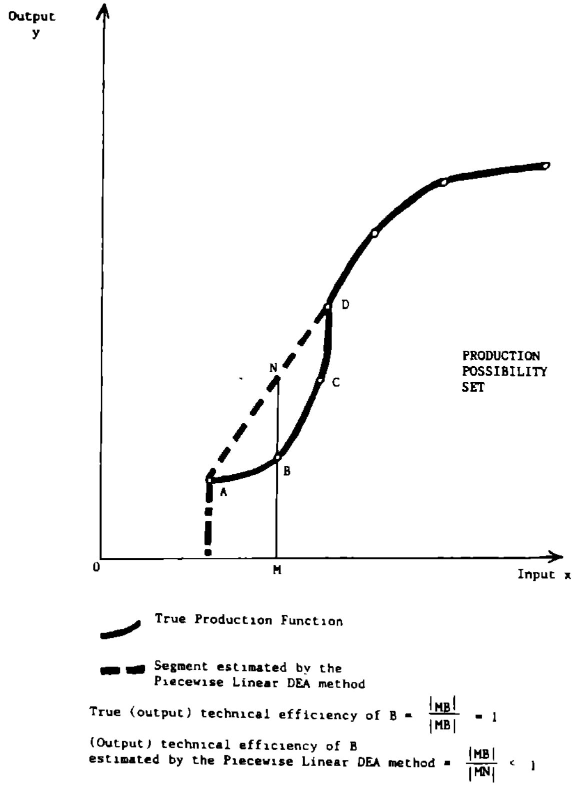

FIGURE 1

reasons provided to motivate such instances of increasing marginal products. See, for instance, Henderson and Quandt (1980, p. 68). To allow for such situations we replace the (ordinary) convexity postulate of BCC by "geometric" convexity to interpolate between observed production possibilities. This implies that instead of the piecewise linear surfaces employed by BCC, we estimate the production correspondence by pieces of loglinear surfaces. Therefore, geometric convexity permits the identification of nonconcavity of the production function between the points A and D in Figure 1, whereas ordinary convexity estimates this region of the production function as the line segment joining the points A and D. If in an empirical application there are *a priori* reasons to believe that marginal products are increasing in some regions, then the loglinear model described here is the appropriate DEA model for the analysis. This modification of the BCC model to allow for increasing (and decreasing)[2] marginal products represents the principal contribution of this paper. It is also of interest to note that in this modified model the estimated production function can be S-shaped, whereas this is not possible in the BCC model.

Banker, Charnes, Cooper, and Schinnar (1981) also describe a procedure for piecewise loglinear estimation of the efficient production surface. In their formulation, the segments are given by systems of s equations of the form $y_r = A_r \prod_{i=1}^{m} x_i^{\alpha_{ir}}$, $r = 1, \ldots s$. Evidently, such a system of equations represents a production situation where the outputs do not compete for the inputs, that is, by assumption $\partial y_r / \partial y_k = 0$ for all $r, k = 1, \ldots s$. Joint production of the outputs meat and hide, from the input cattle, or chemical processes yielding two (or more) products in fixed proportions, such

[2]In fact, the model allows for increasing marginal products at first, and then decreasing marginal products once "overcrowding" sets in and gains from specialization diminish.

as oxygen and hydrogen using the input electric power in the electrolysis of water, are some production situations that will satisfy the requirements of their model in Banker, Charnes, Cooper, and Schinnar (1981). When production is efficient, increases in the oxygen output can only be accomplished in such a situation by increasing the consumption of inputs, and *not* by simply curtailing the production of hydrogen while keeping inputs consumption constant. The piecewise loglinear estimation procedure described in §§2, 3, and 4 is appropriate for joint production processes where $\partial y_r/\partial y_k < 0$, while the procedure described in §5 addresses the production situation considered in Banker, Charnes, Cooper, and Schinnar (1981).

Charnes et al. (1982) employed this loglinear envelopment principle in Banker, Charnes, Cooper, and Schinnar (1981) to suggest a multiplicative efficiency measure. However, this formulation is not consistent with a postulated underlying production technology of the type described in Banker, Charnes, and Cooper (1984) or in this paper. Subsequently, Charnes et al. (1983) have modified their model. But this model differs from the other DEA models, including the three models presented in this paper, inasmuch as it employs a *nonradial* efficiency measure instead of the usual radial Farrell efficiency measure. It must also be noted that the objective of Charnes et al. was the specification of a multiplicative efficiency model and not the examination of the characteristics of the associated production technology. Consequently, their models do not deal with the estimation of returns to scale, most productive scale sizes and aggregate technical and scale efficiencies, addressed in this paper. Furthermore, they do not also address the noncompeting outputs situations considered in Banker, Charnes, Cooper, and Schinnar (1981) and in §5 of this paper.

2. Postulates for a Production Possibility Set

We shall estimate a production possibility set $T = \{(Y,X) \mid \text{the outputs } Y \text{ can be produced from the inputs } X\}$ from observed production data (Y_j, X_j), $j = 1, \ldots n$, for n decision making units (DMUs), where $Y_j \equiv (y_{1j}, \ldots y_{rj}, \ldots y_{sj}) > 0$[3] is a vector of outputs and $X_j \equiv (x_{1j}, \ldots x_{ij}, \ldots x_{mj}) > 0$ is a vector of inputs. We postulate[4] the following properties for the production possibility set T:

Postulate 1. Geometric Convexity. If $(Y_l, X_l) \in T$, $l = 1, \ldots k$, and $\lambda_l \geqslant 0$ are nonnegative scalars such that $\sum_{l=1}^{k} \lambda_l = 1$, then $(Y_0', X_0') \in T$, where $y_{r0}' = \prod_{l=1}^{k} y_{rl}^{\lambda_l}$ and $x_{i0}' = \prod_{l=1}^{k} x_{il}^{\lambda_l}$.

Postulate 2. Monotonicity. (a) If $(Y,X) \in T$, and $X' \geqslant X$, then $(Y,X') \in T$.
(b) If $(Y,X) \in T$, and $0 < Y' \leqslant Y$, then $(Y',X) \in T$.

Postulate 3. Inclusion of Observations. All observed vectors $(Y_j, X_j) \in T$, $j = 1, \ldots n$.

Postulate 4. Minimum Extrapolation. T is the intersection of all sets $\bar{T}$ satisfying Postulates 1, 2, and 3.

The production possibility set T is then given by

$$T = \left\{ (Y,X) \mid 0 < y_r \leqslant \prod_{j=1}^{n} y_{rj}^{\lambda_j},\ r = 1, \ldots s,\ x_i \geqslant \prod_{j=1}^{n} x_{ij}^{\lambda_j} > 0,\ i = 1, \ldots m, \right.$$
$$\left. \text{for some } \lambda_j \geqslant 0 \text{ with } \sum_{j=1}^{n} \lambda_j = 1 \right\} \quad (1)$$

[3]The following conventions are employed for inequality signs relating two k-dimensional vectors:
(i) Vector $A >$ vector B $\Leftrightarrow$ each component $a_i > b_i$, $i = 1, \ldots k$.
(ii) Vector $A \geqslant$ vector $B \Leftrightarrow$ each component $a_i \geqslant b_i$, $i = 1, \ldots k$.

[4]For convenience in exposition, we have divided the minimum extrapolation postulate in Banker, Charnes, and Cooper (1984) into Postulates 3 and 4 here. Note that Postulate 1 represents the only change from the set of postulates used by Banker, Charnes, and Cooper.

where (Y_j, X_j), $j = 1, \ldots n$, are the observed production data for DMUs. We also define a closed and convex set $\hat{T}$ as

$$\hat{T} = \left\{ (\hat{Y}, \hat{X}) \mid \hat{y}_r \leq \sum_{j=1}^{n} \lambda_j \hat{y}_{rj}, \hat{x}_i \geq \sum_{j=1}^{n} \lambda_j \hat{x}_{ij}, \text{ for some } \lambda_j \geq 0 \text{ with } \sum_{j=1}^{n} \lambda_j = 1 \right\} \quad (2)$$

where $\hat{y}_r = \ln y_r$, $\hat{x}_i = \ln x_i$, $\hat{y}_{rj} = \ln y_{rj}$, $\hat{x}_{ij} = \ln x_{ij}$, $\hat{Y} = (\hat{y}_1, \ldots \hat{y}_r, \ldots \hat{y}_s)$ and $\hat{X} = (\hat{x}_1, \ldots \hat{x}_i, \ldots \hat{x}_m)$. We shall use "carets" over scalars to indicate natural logarithms of positive real numbers and "carets" over vectors to indicate natural logarithm of each scalar component. Because of the strict monotonicity of the logarithmic transformation, it follows that there exists a one-to-one mapping of the set $\hat{T}$ onto the production possibility set T.

We shall base our measure of technical inefficiency on a modification of Shephard's (1970) distance function in a manner analogous to the approach adopted by Banker (1980b) and Banker, Charnes, and Cooper (1984). Shephard's output distance measure g_0^* for an output-input vector $(Y_0, X_0) \in T$ is given by:

$$g_0^* = \max g_0 \qquad \text{subject to} \qquad (3)$$

$$g_0 y_{r0} \leq \prod_{j=1}^{n} y_{rj}^{\lambda_j}, \qquad r = 1, \ldots s,$$

$$x_{i0} \geq \prod_{j=1}^{n} x_{ij}^{\lambda_j}, \qquad i = 1, \ldots m,$$

$$1 = \sum_{j=1}^{n} \lambda_j,$$

$$g_0, \lambda_j \geq 0.$$

Shephard's output distance measure g_0^* is equal to one, if and only if the observation (Y_0, X_0) lies on the boundary of the production possibility set T.[5] However, all points on the boundary may not lie in the efficient subset of T.[6] Therefore, to identify the slacks in the $(s + m)$ constraints in (3), we introduce multiplicative coefficients $\psi_{r0} \geq 1$ and $\xi_{i0} \geq 1$, so that

$$\psi_{r0} g_0 y_{r0} = \prod_{j=1}^{n} y_{rj}^{\lambda_j}, \qquad r = 1, \ldots s, \quad \text{and} \qquad (4)$$

$$x_{i0} = \xi_{i0} \prod_{j=1}^{n} x_{ij}^{\lambda_j}, \qquad i = 1, \ldots m.$$

[5]Note that if (Y_0, X_0) is not on the boundary, then g_0^* must be greater than one, and hence, the nonnegativity constraint on g_0 can be modified to read $g_0 \geq 1$ without changing the optimal solution.

[6]See Banker, Charnes, and Cooper (1984, p. 1083) for a detailed discussion.

Taking logarithms, we can next write the following linear programming formulation to measure technical efficiency as in Banker (1980b) and BCC (1984):

$$\hat{z}_0^* = \max \hat{g}_0 + \epsilon\left[\sum_{r=1}^{s} \hat{\psi}_{r0} + \sum_{i=1}^{m} \hat{\xi}_{i0}\right] \quad \text{subject to} \tag{5}$$

$$\hat{g}_0 + \hat{y}_{r0} + \hat{\psi}_{r0} = \sum_{j=1}^{n} \lambda_j \hat{y}_{rj}, \qquad r = 1, \ldots s,$$

$$\hat{x}_{i0} + \hat{\xi}_{i0} = \sum_{j=1}^{n} \lambda_j \hat{x}_{ij}, \qquad i = 1, \ldots m,$$

$$1 = \sum_{j=1}^{n} \lambda_j,$$

$$\hat{g}_0, \hat{\psi}_{r0}, \hat{\xi}_{i0}, \lambda_j \geqslant 0,$$

and $\epsilon > 0$ is a small "non-Archimedean" number.

The optimal value $z_0^* = \exp\{\hat{z}_0^*\}$ obtained from the linear programming formulation in (5) provides us with a measure of the technical inefficiency[7] of the observation $(Y_0, X_0) \in T$.[8] The observation is efficient if and only if $z_0^* = 1$, which in turn is equivalent to $g_0^* = \psi_{r0}^* = \xi_{i0}^* = 1$.

3. Estimation of Production Characteristics

It is evident from the analysis in BCC (1984) that $\sum_{i=1}^{m} \alpha_{i0}^* \hat{x}_i - \sum_{r=1}^{s} \beta_{r0}^* \hat{y}_r + \alpha_{00}^* = 0$ represents a supporting hyperplane for the convex set $\hat{T}$ at a technically efficient point $(\hat{Y}_0, \hat{X}_0)$, where $\alpha_{i0}^*, \alpha_{00}^*, \beta_{r0}^*$ are the optimal values of the variables in the following dual of the linear program in (5):

$$\min \sum_{i=1}^{m} \alpha_{i0}\hat{x}_{i0} - \sum_{r=1}^{s} \beta_{r0}\hat{y}_{r0} + \alpha_{00} \quad \text{subject to} \tag{6}$$

$$\sum_{i=1}^{m} \alpha_{i0}\hat{x}_{ij} - \sum_{r=1}^{s} \beta_{r0}\hat{y}_{rj} + \alpha_{00} \geqslant 0, \qquad j = 1, \ldots n,$$

$$\sum_{r=1}^{s} \beta_{r0} = 1,$$

$$\alpha_{i0}, \beta_{r0} \geqslant \epsilon > 0, \qquad i = 1, \ldots m, \quad r = 1, \ldots s,$$

ϵ is a small "non-Archimedean" number.

If the optimal values α_{i0}^*, α_{00}^* and β_{r0}^*, are uniquely determined from the linear programming problem in (6), then there exists a unique supporting hyperplane at $(\hat{Y}_0, \hat{X}_0)$. Therefore, it follows that

$$\prod_{r=1}^{s} y_r^{\beta_{r0}^*} = e^{\alpha_{00}^*} \prod_{i=1}^{m} x_i^{\alpha_{i0}^*} \tag{7}$$

is a surface bounding the set T at the technically efficient point (Y_0, X_0). Further, the

[7]It is evident that the Archimedean component of this measure is independent of the choice of units of measurement. This invariance property follows immediately by appropriate substitutions in the formulation in (5).

[8]If any of the inputs or outputs are not controllable by the DMU manager, then this formulation may be modified as in Banker and Morey (1983).

efficient production surface of T consists of piecewise segments of surfaces[9] represented by equations in (7) corresponding to different efficient points (Y_0, X_0) in T.

The production characteristics locally at an efficient point (Y_0, X_0) are then obtained directly from the equation of the bounding surface at that point. Thus, we have[10]

(8a) rate of substitution of input i for input $l = -\partial x_l/\partial x_i = (\alpha_{i0}^*/x_{i0})/(\alpha_{l0}^*/x_{l0})$,

(8b) rate of transformation of output r for output $k = -\partial y_k/\partial y_r = (\beta_{r0}^*/y_{r0})/(\beta_{k0}^*/y_{k0})$,

(8c) marginal product of input i for output $r = \partial y_r/\partial x_i = (\alpha_{i0}^*/x_{i0})/(\beta_{r0}^*/y_{r0})$,

(8d) increasing, constant or decreasing returns to scale prevail at (Y_0, X_0) depending on whether $\sum_{i=1}^{m} \alpha_{i0}^*$ is greater than, equal to or less than one.

For the corner points at the intersection of two or more of the bounding surface segments, these production characteristics are not determined uniquely, and the rates of substitution and transformation and the marginal products will not, in general, be continuous functions. In Banker and Maindiratta (1983), we describe a procedure for determining the upper and lower bounds for the estimated values of these measures at such points of discontinuity.

Finally, we turn to the estimation of the most productive scale size (mpss) for specific input and output mixes. Banker (1984) defines the concept of most productive scale size. A production possibility $(Y_M, X_M) \in T$ represents mpss for its specific mix of inputs and outputs if and only if for all $(\mu Y_M, \omega X_M) \in T$ we have $\mu \leqslant \omega$. We formulate the following linear programming problem to determine the mpss for a specific input and output mix represented by $(Y_0, X_0) \in T$:

$$\min \hat{\omega} - \hat{\mu} \qquad \text{subject to} \tag{9}$$

$$\hat{\mu} + \hat{y}_{r0} \leqslant \sum_{j=1}^{n} \lambda_j \hat{y}_{rj}, \qquad r = 1, \ldots s,$$

$$\hat{\omega} + \hat{x}_{i0} \geqslant \sum_{j=1}^{n} \lambda_j \hat{x}_{ij}, \qquad i = 1, \ldots m,$$

$$1 = \sum_{j=1}^{n} \lambda_j,$$

$$\lambda_j \geqslant 0, \qquad \hat{\mu}, \hat{\omega} \text{ are unconstrained in sign.}$$

The production possibility $(\mu^* Y_0, \omega^* X_0)$ then represents mpss for the input and output mix (Y_0, X_0). As in the axiomatic approach of Banker, Charnes, and Cooper (1984), we can axiomatically derive the formulation in (9) by adding the "Ray Extension" postulate[11] to the set of postulates described earlier in §2. We can then also

[9]Such parametric forms have been used, for instance, by Klein (1947) and Theil (1980). However, our approach provides a possibly more flexible estimation procedure because instead of being constrained to fitting a single parametric form over the entire range of observations, several segments of such surfaces are used to closely envelope the production possibility set. It is also interesting to note that since T is not convex in this case, some points on its technically efficient surface may not be allocatively efficient for any set of prices.

[10]For simplicity in exposition, partial derivatives are used to denote these notions It must be noted, however, that the partial derivatives will not be defined at the points at the intersection of two or more of the bounding surface segments. These notions can be defined rigorously even for these instances in terms of subgradients. For instance, we can define ξ_{il} to be a rate of substitution of input i for input l at an efficient point (Y_0, X_0) if and only if for all points $(Y_0, X_1) \in T$ with $x_{k1} = x_{k0}$ for $k \neq l, i$, and $x_{k1} = x_{k0} + \Delta_k$ for $k = l, i$, we have $\Delta_l > -\xi_{il}\Delta_i$.

[11]Ray Extension postulate requires that $(Y, X) \in T$, $t > 0 \Rightarrow (tY, tX) \in T$.

obtain $\omega^*/\mu^* = \exp\{\hat{\omega}^* - \hat{\mu}^*\} \leqslant 1$ as the aggregate technical and scale efficiency measure, as in Banker, Charnes, and Cooper (1984), and decompose it into technical efficiency and scale efficiency measures.

4. A Simulation Study

The piecewise loglinear model described in this paper was employed to estimate the production frontier from input and output data randomly generated from a known technology. The simulation data in our study were the same as the data employed by Banker, Charnes, Cooper, and Maindiratta (1986) in their simulation study comparing the DEA models in Banker, Charnes and Cooper (1984) with a COLS translog estimation model.

The known technology for one output and two inputs was specified in terms of its efficient production function $Q = f(K, L)$ where:

$$Q = \begin{cases} 0.631 \quad K^{0.65} \quad L^{0.55} & \text{for} \quad 5 \leqslant K \leqslant 10, \quad 5 \leqslant L \leqslant 10, \\ 0.794 \quad K^{0.65} \quad L^{0.45} & \text{for} \quad 5 \leqslant K \leqslant 10, \quad 10 \leqslant L \leqslant 15, \\ 1.259 \quad K^{0.35} \quad L^{0.55} & \text{for} \quad 10 \leqslant K \leqslant 15, \quad 5 \leqslant L \leqslant 10, \\ 1.585 \quad K^{0.35} \quad L^{0.45} & \text{for} \quad 10 \leqslant K \leqslant 15, \quad 10 \leqslant L \leqslant 15. \end{cases} \tag{10}$$

The actual output level Y was derived from the expression $Y = eQ$, where the actual output efficiency measure e was generated from the probability distribution defined by:

$$\Pr\{e = 1\} = 0.3, \qquad \Pr\{0.65 \leqslant e < 1\} = 0.7 \tag{11}$$

with the specific values of e when $0.65 \leqslant e < 1$ being obtained from the uniform distribution on the interval [0.65, 1). The values of K and L were randomly generated from the uniform probability distribution over the interval [5, 15]. Two samples of 500 and 100 observations were obtained.

The estimation results[12] using the piecewise loglinear model in (6) were generally superior to even the excellent results obtained using the piecewise linear models described in Banker, Charnes, and Cooper (1984). See Table 1. These results are also apparent from the cumulative distributions for the absolute deviations plotted in Figure 2. The estimates for rates of substitution between the two inputs obtained from the piecewise loglinear model were also closer to the actual values of this production characteristic than the estimates obtained from the piecewise linear model. See Table 3.

The fact that the piecewise loglinear model described in this paper performs even better than the piecewise linear model described in Banker, Charnes, and Cooper (1984) is not surprising because the underlying production function exhibits nonconcavity in some regions. These regions correspond to $5 \leqslant K < 10$, where the marginal product is increasing along any specific ray of input mix. It is also evident from comparing Tables 1 and 2 that the difference between the accuracy of the two methods is even larger for this region of increasing marginal products.

These results were further accentuated when the simulation study was replicated for production functions with greater degree of increasing marginal products. These simulations, therefore, indicate that if we do have some *a priori* information about the nature of the underlying production technology, then this knowledge can be exploited by choosing the appropriate variation of the DEA models.

[12]The value of ϵ used in the estimation was $\epsilon = 10^{-8}$.

TABLE 1

Average Absolute Difference Between Estimated and Actual Technical Efficiencies

	Mean		Median	
	$n = 500$	$n = 100$	$n = 500$	$n = 100$
DEA piecewise linear	0.0032	0.0100	0 0003	0.0017
Translog	0.0237	0.0444	0.0225	0.0475
DEA piecewise loglinear	0.0022	0.0089	0.00004	0.0001

TABLE 2

Average Absolute Difference Between Estimated and Actual Technical Efficiencies for Observations with $K < 10$

		$n = 500$	$n = 100$
Number of observations with $K < 10$		247	42
DEA piecewise linear	mean	0.0041	0.0146
	median	0.0005	0.0009
DEA piecewise loglinear	mean	0.0024	0.0136
	median	0.00003	0.00005

TABLE 3

Average Absolute Percentage Difference Between Estimated and Actual Rates of Substitution

	Mean		Median	
	$n = 500$	$n = 100$	$n = 500$	$n = 100$
DEA piecewise linear	8.47	15.19	4.03	7.15
Translog	18 07	24.27	17 62	22.65
DEA piecewise loglinear	3.94	13 25	0.007	0.83

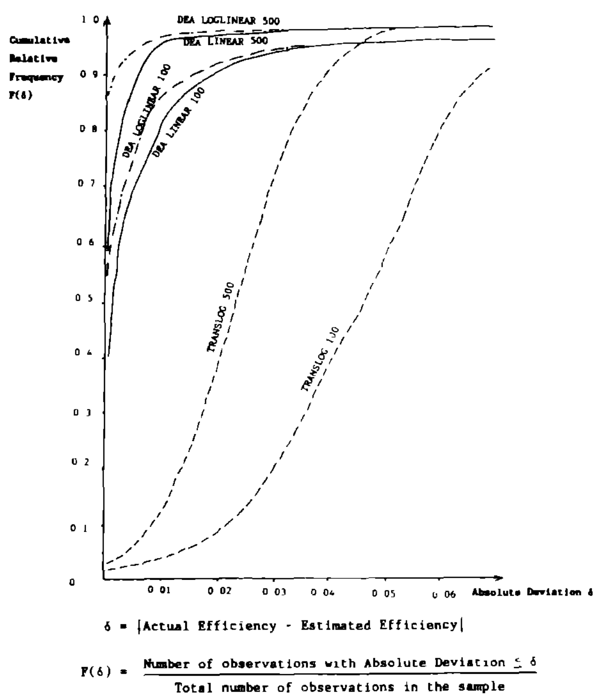

FIGURE 2. Cumulative Distributions for Absolute Differences Between Estimated and Actual Efficiencies

5. Production Technologies with Noncompeting Outputs

In the earlier sections we dealt with production technologies for which it was assumed that $\partial y_r/\partial y_k < 0$ in general, and the production correspondence between inputs and outputs had to be estimated simultaneously for all outputs. Banker, Charnes, Cooper, and Schinnar (1981) described a bi-extremal procedure for piecewise loglinear estimation of the efficient production surface for situations in which the outputs do not compete with each other for inputs. That is, by assumption $\partial y_r/\partial y_k = 0$, and a separate production function $y_r = A_r \prod_{i=1}^{m} x_i^{\alpha_{ir}}$, $r = 1, \ldots s$, was estimated for each output r. Hence, in their model, for a given input vector, the efficient levels of different outputs were estimated independent of each other. The efficiency of individual observations was measured in two stages. First, loglinear envelopes were estimated to determine augmented output values, and next these augmented output values were employed as referent production possibilities to measure the efficiency of the original observation. Here, we shall develop a rigorous estimation procedure that is consistent with the postulated properties of such production technologies with noncompeting outputs. We shall provide a one-stage procedure for estimating the production surface segments and technical efficiencies.

The assumed noncompeting nature of the output requires that our earlier Postulate 2(b) be modified to be:

Postulate 2. Monotonicity. (b) If (Y, X), $(Y', X) \in T$, then $(Y'', X) \in T$, where $Y'' \equiv (y_1'', \ldots y_s'') > 0$ and each $y_r'' \leqslant \max\{y_r, y_r'\}$.

The modified production possibility set is then given by

$$T = \left\{ (Y, X) \,|\, \text{for each } r, \text{ there exist some } \theta_{rj} \geqslant 0 \text{ with } \sum_{j=1}^{n} \theta_{rj} = 1, \right.$$
$$\left. 0 < y_r \leqslant y_r^* = \prod_{j=1}^{n} y_{rj}^{\theta_{rj}},\ r = 1, \ldots s,\ x_i \geqslant \prod_{j=1}^{n} x_{ij}^{\theta_{rj}} > 0,\ i = 1, \ldots m \right\}. \quad (12)$$

Proceeding as before, the technical efficiency measure for an observation (Y_0, X_0) is obtained from the following linear program:

$$\hat{z}_0^* = \max \sum_{r=1}^{s} \hat{g}_{r0} + \epsilon \sum_{i=1}^{m} \hat{\xi}_{i0} \qquad \text{subject to} \qquad (13)$$

$$\hat{g}_{r0} + \hat{y}_{r0} = \sum_{j=1}^{n} \theta_{rj} \hat{y}_{rj}, \qquad r = 1, \ldots s,$$

$$\hat{x}_{i0} - \hat{\xi}_{i0} \geqslant \sum_{j=1}^{n} \theta_{rj} \hat{x}_{ij}, \qquad i = 1, \ldots m \quad \text{and} \quad r = 1, \ldots s,$$

$$1 = \sum_{j=1}^{n} \theta_{rj}, \qquad r = 1, \ldots s,$$

$$\hat{g}_{r0}, \hat{\xi}_{i0}, \theta_{rj} \geqslant 0$$

and $\epsilon > 0$ is a small "non-Archimedean" number.

The natural logarithm of the technical efficiency measure is given by $\hat{z}_0^*$. The characteristics of the efficient production surface are estimated as before by considering the dual of the above linear program.

This extension to noncompeting outputs situation provides a good way to conclude this paper because it illustrates how our general axiomatic approach can be employed with modified postulates to reflect different *a priori* beliefs about the production technology.[13]

[13]Helpful comments and suggestions by two anonymous referees are gratefully acknowledged

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