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Continuous Optimization

A simulation study of DEA and parametric frontier models in the presence of heteroscedasticity

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Abstract

This paper studies the effects of heteroscedasticity on the following five types of estimators: (1) Data Envelopment Analysis (DEA) per se as well as DEA joined to regression forms, (2) Corrected Ordinary Least Squares based on maximum residual (COLS-R), (3) Corrected Ordinary Least Squares based on moments of residuals (COLS-M), (4) Maximum Likelihood Estimation (MLE), and (5) Goal Programming with one-sided deviations as in Aigner and Chu (A&C). This is accomplished with simulated data in an experiment designed around a single output–single input production function which is piecewise Cobb–Douglas. Robustness of results is confirmed with another experiment employing a shifted smooth Cobb–Douglas production function. The model has a composed error term consisting of two components—one for measurement error and the other for inefficiency. The simulation results indicate that heteroscedasticity does not have an adverse impact on DEA-based estimators and that DEA-based estimators are the best estimators of efficient output even under heteroscedasticity.

Keywords: Production function; Inefficiency estimators; Heteroscedasticity; Simulation study

1. Introduction

Using a composed error model, with two components, Bojanic et al. (1998, hereafter BCF) examine small sample properties of frontier estimators with heteroscedasticity present in the two-sided measurement error component but not in the one-sided component used to represent inefficiency. See expression (4) and subsequent discussion in Bojanic et al. As noted in Banker et al. (2002), the BCF study did *not* include Data Envelopment Analysis (DEA) estimators. Instead, they used only the Aigner and Chu (1968, hereafter A&C) parametric frontier model but erroneously labeled it as DEA. However, as also noted in Banker,

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Chang and Cooper, the Bojanic et al. study did point up the need for extending previous research in DEA to comprehend the effects of heteroscedasticity on DEA estimators. This is accomplished in the present paper by means of Monte Carlo experiments in which we extend the studies by BCF, and others, beyond the case of two-sided measurement errors in order to examine the effects of heteroscedasticity on the one-sided error term associated with the inefficiencies that are also involved in stochastic approaches to performance evaluations.

Heteroscedasticity arises when variances of regression disturbances are not constant across observations (Greene, 2000, p. 499). Banker (1993) showed that DEA estimators are consistent for any distribution of inefficiency satisfying the envelopment (one-sided error) and the likelihood of efficient performance postulates. Since the presence of heteroscedasticity does not violate these two postulates, the consistency property of DEA estimators holds under heteroscedasticity when only one-sided error is present. In addition, since OLS coefficient estimates are unbiased under heteroscedasticity if the regressors and the error term are uncorrelated (Greene, 2000, p. 457), all corrected ordinary least squares (COLS) coefficient estimates *except the intercept term* are also unbiased in the presence of heteroscedasticity. However, because the COLS estimator of the intercept term is obtained by shifting its initial estimator based on the maximum value of the OLS residuals (COLS-R) or based on the estimated variance of the heteroscedastic error term (COLS-M), the estimator of the inefficiency measure using the COLS method may be biased. In the case of both maximum likelihood (ML) and A&C estimators, it is not clear why heteroscedasticity should have an effect. While Bojanic et al. (1998) presented Monte Carlo evidence suggesting that both maximum likelihood and COLS estimators outperformed A&C estimators under heteroscedasticity in the two-sided error term, they did not present any conceptual argument to motivate such a finding. In any case, no study has investigated the impact of heteroscedasticity on the comparative performance of the DEA, COLS-R, COLS-M, MLE and A&C estimators.

Our simulation study first considers two cases for the one-sided error term representing inefficiency. In the first case, we consider homoscedastic one-sided error and in the second case, we introduce heteroscedasticity proportional to the input [1]. We turn next to the "composed error" case where the error term has two components, a two-sided component representing measurement error and a one-sided component representing inefficiency. In a Monte Carlo study of the comparative performance of COLS and DEA estimators, Banker et al. (1993) reported that neither the COLS estimator nor the DEA estimator performs well for high measurement errors. We therefore focus on moderate levels of measurement error in this study. In the composed error setting, we first consider heteroscedasticity only in the inefficiency component, then only in the measurement error component, and finally in both the inefficiency and the measurement error components.

The rest of this paper is organized as follows. Section 2 describes our data generating process. Section 3 presents the production function estimators employed in this study. Section 4 reports and discusses the results of the simulation. Section 5 concludes the paper.

2. Design of experiment

For this study, we consider an underlying production function with one output, one input and moderate levels of measurement error. Thus we differ from Bojanic et al. in that we move from their use of a cost function (with one independent variable) to a production function (with one independent variable). This

[1] Our results remain similar when the heteroscedasticity in either the one-sided or the two-sided error (or both) is proportional to a variable independent of the input. Following the suggestion of a referee, we do not report these results here to simplify the paper and reduce the number of tables by one-half.

allows us to focus on technical inefficiency without the need for access to cost and price information. We do this, however, with a function that is more complicated than the one employed by Bojanic et al. Inefficiencies are generated from specified distribution functions. A moderate degree of heteroscedasticity is introduced into both the inefficiencies and the measurement component of the composed error term.

2.1. Production technology

The production technology we use is specified in terms of an efficient production function $y^* = g(x)$, where x, a scalar, is the input, and y^*, another scalar, is the efficient level of output generated from the input x. Specifically, following Banker et al. (1996, 1993, 1987), we represent $g(x)$ by the following piecewise Cobb–Douglas production function [2]:

$$y^* = g(x) = \begin{cases} a_1 x^{b_1}, 10 \leqslant x \leqslant 12, \\ a_2 x^{b_2}, 12 \leqslant x \leqslant 14, \\ a_3 x^{b_3}, 14 \leqslant x \leqslant 16, \\ a_4 x^{b_4}, 16 \leqslant x \leqslant 18, \\ a_5 x^{b_5}, 18 \leqslant x \leqslant 20. \end{cases} \tag{1.1}$$

The a and b coefficients determine the technology for which we use

$$b_1 = 0.8, \quad b_2 = 0.7, \quad b_3 = 0.6, \quad b_4 = 0.5, \text{ and } b_5 = 0.4 \text{ with } a_1 = 10. \tag{1.2}$$

Values of the remaining a's are selected to ensure continuity of the production function. We generate the values of input, x, from the uniform distribution over the interval [10,20].

2.2. Composed error

The error term, ε, as assumed in, say, COLS or ML estimation, is decomposed into two components as follows:

$$\varepsilon = v - \mu. \tag{2}$$

Here $\mu \geqslant 0$ represents inefficiencies and v, which is unconstrained in sign, represents measurement error. Thus, for our efficient production function $y^* = g(x)$ we have

$$\ln y = g(x) + \varepsilon = g(x) + v - \mu, \tag{3}$$

where $\ln y$, the logarithm of the observed value, need not be efficient, whereas the $y^* = g(x)$ values are efficient as required by economic theory. The inefficiency term, $\mu \geqslant 0$, therefore, reflects the output shortfalls due to inefficiency. The error term, v, reflects all other types of error and is not constrained in its sign.

2.3. Homoscedastic cases

Our inefficiencies, like our measurement errors, are generated from specified probability distributions. Measurement errors are generated from the normal probability distribution with $\sigma_v^2 = 0.004$. That is, we specify measurement errors with a moderate variance in a manner similar to the moderate measurement error case in Banker et al. (1993). Inefficiency values are generated from the exponential distribution with

[2] Our basic result that heteroscedasticity has no adverse impact on the DEA estimators of inefficiency remains robust to simpler smooth parametric representations of $g(x)$. See the sensitivity analysis in Section 5.4.

variance $\sigma_\mu^2 = 0.02$. This specification is similar to the one used in Banker et al. (1993). It is also similar to the middle case used in Bojanic et al. (1998) in that $\sigma_\mu^2 = 5\sigma_v^2$. For the exponential distribution with parameter λ we have $\sigma_\mu^2 = (1/\lambda^2)$. The mean value of these inefficiencies is obtained from

$$E(\mathrm{e}^\mu) = \int_0^\infty \mathrm{e}^\mu \lambda \mathrm{e}^{-\lambda\mu}\,\mathrm{d}\mu = \lambda \int_0^\infty \mathrm{e}^{(1-\lambda)\mu}\,\mathrm{d}\mu = \frac{\lambda}{\lambda - 1}. \tag{4}$$

As just noted, we are assuming $\sigma_\mu^2 = 0.02$ for the inefficiency distribution in this (the homoscedastic) case, so that $\lambda^2 = 50$. Hence, we have

$$\frac{\lambda}{(\lambda - 1)} = \frac{\sqrt{50}}{\sqrt{50} - 1} = \frac{7.07106}{6.07106} = 1.16.$$

Thus the expected value of the generated inefficiencies is 16%. This, too, is very similar to Banker et al. (1993).

2.4. Degree of heteroscedasticity

We consider six different cases with heteroscedasticity present in the inefficiency component, the measurement error component, or both components of the composed error.

These six cases considered in our simulation experiments can be summarized as follows:

		One-sided (inefficiency) error term	
		Homoscedasticity	Heteroscedasticity
Two-sided (measurement) error term	No measurement error	Case 1	Case 2
	Homoscedasticity	Case 3	Case 4
	Heteroscedasticity	Case 5	Case 6

To focus on the previously unexplored case of heteroscedasticity in the distributions used to generate the $\mu \geqslant 0$, we start with two cases, as follows, in which $\sigma_v^2 = 0$ enables us to focus only on the inefficiency component.

Case 1. Heteroscedasticity is not present in the inefficiencies. This is our base-line case.

Case 2. Heteroscedasticity in the inefficiencies is made proportional to input via the relation

$$\sigma_{\mu_j}^2 = k_\mu x_j^2. \tag{5.1}$$

Similar to Bojanic et al. (1998), we specify only a moderate level of heteroscedasticity for which we use

$$\sigma_{\mu_j}^2 = k_\mu x_j^2 \text{ s.t. } E(k_\mu x_j^2) = 0.02. \tag{5.2}$$

Since input x is generated from the uniform distribution defined over the interval [10,20], the value of k_μ is obtained from

$$k_\mu = \frac{0.02}{E(x_j^2)} = \frac{0.02}{\int_{10}^{20} \frac{x^2}{10}\,\mathrm{d}x} = \frac{0.02}{\frac{7000}{30}} = 0.0000857. \tag{5.3}$$

We next relax the condition $\sigma_v^2 = 0$ so both components of the composed error are present. To continue with our focus on heteroscedasticity in the inefficiencies, however, we do not introduce heteroscedasticity in the measurement error term in the next two cases, which are:

Case 3. In a manner analogous to Case 1 we do not introduce heteroscedasticity in the distributions used to generate measurement errors and the inefficiencies.

Case 4. As in Case 2, we make heteroscedasticity proportional to the input

$$\sigma^2_{\mu_j} = k_\mu x_j^2, \tag{6}$$

with $k_\mu = 0.0000857$, as in Case 2.

Next, we consider the case in which heteroscedasticity is present only in the measurement error component (and not in the inefficiency component) of the composed error.

Case 5. Heteroscedasticity in the distributions used for measurement error, v, is made proportional to input x via the relation

$$\sigma^2_{v_j} = k_v x_j^2, \tag{7.1}$$

where k_v is determined from the relation

$$\sigma^2_{v_j} = k_v x_j^2 \text{ s.t. } E(k_v x_j^2) = 0.004. \tag{7.2}$$

With x generated from the previously described uniform distribution over the interval [10, 20], this gives $k_v = 0.0000171$.

Finally, we consider the case in which heteroscedasticity is present in both the inefficiency component and the measurement error component of the composed error.

Case 6. Heteroscedasticity in both the inefficiency component and the measurement error component is made proportional to input x via the relations as in Case 4 and Case 5, respectively.

As we will see, these six cases produce quite a number of possibilities to consider. Hence we did not deem it prudent to proceed further at this moment into still other ways in which heteroscedasticity might be present. We can therefore move our discussion of the experimental design by turning to the manner in which the variables μ, v and y were generated.

To obtain the inefficiencies we used the relations $\sigma^2_{\mu_j} = 0.02$ when no heteroscedasticity is present or $\sigma^2_{\mu_j} = k_\mu x_j^2$ when heteroscedasticity is present to designate the exponential distribution for use in generating the $\mu_j \geqslant 0$ values. Thus, for each sample of size n we used an exponential distribution to generate n values of inefficiencies. In a similar manner we used the relations $\sigma^2_{v_j} = 0.004$ when no heteroscedasticity is present or $\sigma^2_{v_j} = k_v x_j^2$ when heteroscedasticity is present to designate the normal distribution for use in generating the v_j values we use.

2.5. Output values y_j

Each x_j value as generated from the uniform distribution [10,20] was substituted in the appropriate segment of (1.1) and (1.2) to obtain the corresponding y_j^* value. By the definition of a "production function" these y_j^* all represent efficient output values. We therefore adjusted these values to obtain the observed output values y_j that reflect inefficiencies and measurement errors by means of the following formula:

$$y_j = y_j^* \frac{e^{v_j}}{e^{\mu_j}}. \tag{8}$$

Thus, although our inefficiency and measurement error components are relatively small, they are also pervasive. Hence we have tried to provide a series of tests to help us discriminate between our different estimating models according to how well they behave under each of the six cases. For this purpose we used three different sample sizes—25, 50 and 100—and replicated each case 100 times to yield 1800 $(= 100 \times 6 \times 3)$ samples involving 105,000 simulated observations which were all treated in the manner we have now described in order to examine the relative performances of the estimating models (and methods) we employed to estimate the true (= efficient) values obtained from (1.1) and (1.2).

3. Production function estimation methods

In this section, we describe the five different types of estimation models we studied and the criteria we used to evaluate their comparative performances.

3.1. Estimation models

The five types of estimation models are DEA and regression-fitted DEA, the Maximum Residual method of Corrected Ordinary Least Squares (COLS-R), the Moment method of Corrected Ordinary Least Squares (COLS-M), Maximum Likelihood Estimation (MLE), and Aigner and Chu (A&C) method.

3.1.1. DEA estimation models

The DEA model we used is the BCC model (Banker et al., 1984) which was given the following "output oriented" form:

$$\begin{aligned} \hat{\theta}_j \equiv \text{Max} \quad & \theta_j \\ \text{s.t.} \quad & \sum_{k=1}^{n} \lambda_k y_k \geqslant \theta_j y_j, \\ & \sum_{k=1}^{n} \lambda_k x_k \geqslant x_j, \\ & \sum_{k=1}^{n} \lambda_k = 1, \\ & \theta_j, \lambda_k \geqslant 0. \end{aligned} \tag{9}$$

Thus, the objective is to maximize θ_j subject to these constraints—as generated for each sample—with y_j and x_j representing the observed values for the DMU_j to be evaluated relative to all of the observations in this sample. Banker (1993) has shown that the BCC model of DEA provides a consistent estimator under any distribution of inefficiency satisfying the envelopment and the likelihood of efficient performance postulates.

DEA is non-parametric. Hence to compare its parameter estimation capabilities with the parameter estimates obtained from the other four methods—COLS-R, COLS-M, MLE and A&C—we also joined the above DEA model to the following log-linear and log-quadratic regression models in a manner similar to Farrell (1957):

$$\text{Log-linear (LL) fitted DEA model} : \ln(y_j\hat{\theta}_j) = \beta_0 + \beta_1 \ln x_j + \varepsilon_j, \tag{10}$$

$$\text{Log-quadratic (LQ) fitted DEA model} : \ln(y_j\hat{\theta}_j) = \beta_0 + \beta_1 \ln x_j + \beta_{11}(\ln x_j)^2 + \varepsilon_j. \tag{11}$$

Here θ_j represents the optimal value obtained from (9) for observation j. This optimal θ_j is applied to the observed y_j values for DMU_j to obtain the upward adjustment given by $\hat{\theta}_j y_j \geqslant y_j$ and provide an estimate of the efficient output.

These $\hat{\theta}_j$ values represent optimal estimates for *each* observation. However, the least squares calculation produces an estimated production function with parameter values that are optimized over *all* observations in the sample. Hence the estimates of β_0, β_1 and β_{11} resulting from (10) and (11) represent "average" values of these parameters taken over all of the thus adjusted "efficient" data.

3.1.2. COLS—maximum residual (COLS-R)

The second type of estimation method we used is the COLS-R method. To estimate the production function from simulated data using the COLS-R, we specified the following log-linear and log-quadratic (translog) models:

$$\text{Log-linear (LL) model}: \ln y_j = \beta_0 + \beta_1 \ln x_j + \varepsilon_j, \tag{12}$$

$$\text{Log-quadratic (LQ) model}: \ln y_j = \beta_0 + \beta_1 \ln x_j + \beta_{11}(\ln x_j)^2 + \varepsilon_j. \tag{13}$$

The COLS-R method proceeds in two steps. In step 1, the OLS procedure is performed. If the functions in (12) and (13) represent the true functional relationship between y and x, then the OLS estimation for these functions results in unbiased estimators of the coefficients except the intercept term. Therefore, in step 2 of the COLS-R method the intercept term is adjusted to obtain (efficient) frontier estimates. For this purpose we corrected the intercept term by shifting the regression upward using the maximum residual value in order to derive an estimated production frontier (Richmond, 1974). This method yields a consistent estimator of the frontier value (Greene, 1980).

3.1.3. COLS—moments of residuals (COLS-M)

The third type of estimation method we employed is the COLS-M method which is identical to the second one except that now the intercept term is shifted upward by the estimated bias of the error term ε (Olson et al., 1980). This is accomplished via the estimation of the second and third moments of OLS residuals as follows.

If $\ln y = \beta_0 + \beta_1 \ln x + \varepsilon$, where $\varepsilon = v - \mu$, $v \sim N(0, \sigma_v^2)$ and $\mu \sim \exp(1/\lambda)$, then the bias of the intercept term is $E(\varepsilon) = E(\mu) = (1/\lambda)$. We can estimate λ consistently using the second moment (μ_2') and the third moment (μ_3'):

$$\mu_2' = E[\varepsilon - E(\varepsilon)]^2 = E\left[\varepsilon^2 - 2\frac{\varepsilon}{\lambda} + \frac{1}{\lambda^2}\right] = \sigma_v^2 + \frac{1}{\lambda^2}, \tag{14}$$

$$\mu_3' = E[\varepsilon - E(\varepsilon)]^3 = E\left[\varepsilon - \frac{1}{\lambda}\right]^3 = -\frac{2}{\lambda^3}. \tag{15}$$

From (15) we have $\hat{\lambda} = (-2/\mu_3')^{(1/3)}$. Therefore, the corrected intercept term using the COLS-M method is $\hat{\beta}_0 + (1/\hat{\lambda}) = \hat{\beta}_0 + (-2/\mu_3')^{(1/3)}$). If the estimated $\mu_3' > 0$ then $1/\mu_3'$ is set equal to zero and the corrected β_0 equals $\hat{\beta}_0$.

3.1.4. Maximum likelihood estimation (MLE) method

The fourth type of estimation method we employed is the MLE method. Recall from Section 2 that the error term $\varepsilon = v - \mu$ is specified as being composed of two independent components, v and μ. Since the symmetric measurement error v is distributed as $N(0, \sigma_v^2)$ and the one-sided inefficiency μ is distributed independently of v and derived from an exponential distribution with variance σ_μ^2, the density function of the composed error term $\varepsilon = v - \mu$ is given by (Meeusen and Broeck, 1977)

$$f(\varepsilon) = \frac{1}{\sigma_\mu}\left[1 - F\left(\frac{\varepsilon}{\sigma_v} + \frac{\sigma_v}{\sigma_\mu}\right)\right] \exp\left[\frac{\varepsilon}{\sigma_\mu} + \frac{\sigma_v^2}{2\sigma_\mu^2}\right], \tag{16}$$

where $F(\cdot)$ is the cumulative distribution function of the standard normal distribution.

The ML estimators of the log-linear and log-quadratic production function models specified in (12) and (13) are obtained from simulated data by the maximization of the log-likelihood function based on (16).

3.1.5. Aigner and Chu (A&C) estimation models

The fifth type of estimation method we used is the Goal Programming type as A&C model with one-sided deviations (Banker et al., 2002). Schmidt (1976) showed that the A&C linear programming model is equivalent to a maximum likelihood estimation model if errors are assumed to have a one-sided exponential distribution. This model, which can be described as a "goal programming model", is specified for the log-linear production function as

$$\begin{aligned} \text{Min} \quad & \sum_{j=1}^{n} \varepsilon_j, \\ \text{s.t.} \quad & \beta_0 + \beta_1 \ln x_j - \ln y_j \leqslant \varepsilon_j, \quad j = 1, \ldots, n, \\ & \varepsilon_j, \beta_0, \beta_1 \geqslant 0. \end{aligned} \tag{17}$$

The criterion of the minimization of the one-sided deviations $\varepsilon_j \geqslant 0$ was used to estimate β_0 and β_1 for each sample.

The linear programming formulation for the log-quadratic production function is

$$\begin{aligned} \text{Min} \quad & \sum_{j=1}^{n} \varepsilon_j \\ \text{s.t.} \quad & \beta_0 + \beta_1 \ln x_j + \beta_{11} (\ln x_j)^2 - \ln y_j \leqslant \varepsilon_j, \\ & \quad j = 1, \ldots, n, \\ & \varepsilon_j, \beta_0, \beta_1, \beta_{11} \geqslant 0. \end{aligned} \tag{18}$$

The coefficient estimates from all of the above models are compared to the true values of the parameters specified for (1.1) and (1.2) and their deviations noted in the absolute value measure we shall shortly describe.

4. Performance criteria

In this study, we evaluated the effects of heteroscedasticity on DEA, COLS-R, COLS-M, ML and A&C estimators and their comparative performance by using mean absolute deviation and median absolute deviation measures for both efficient output and slope. That is, we used these absolute value measures to determine the magnitudes of the deviations between the estimates and the true values (as obtained from substitution in (1.1) and (1.2)). Thus, we calculated the mean absolute deviations (MAD) of efficient output as

$$\text{MAD of output} = \sum_{j=1}^{n} \left| g(x_j) - \hat{y}_j \right| / N = \sum_{j=1}^{n} \left| y_j^* - \hat{y}_j \right| / N, \tag{19}$$

where y_j^* is the efficient output level determined from (1.1) and (1.2) and $\hat{y}_j$ is the value obtained from one of the estimation models we are studying. [3]

[3] In keeping with the usual composed error approach we are thus ignoring input inefficiencies and assuming that all inefficiencies are reflected in the output values. The assumption also appears in the composed error approaches used in stochastic frontier regressions. See the discussion in Bardhan et al. (1998). We might also note that non-zero output slack cannot appear in an optimum for (9) in the single output case. See Brockett et al. (2001) for a proof.

In a similar manner we calculated the mean absolute deviations of our slope estimates from the following formula:

$$\text{MAD of slope} = \sum |\gamma_j - \hat{\gamma}_j|/N, \tag{20}$$

where γ_j is the actual slope calculated from (1.1) and (1.2) as

$$\text{slope } (\gamma_j) = \partial g(x_j)/\partial x_j = b_r g(x_j)/x_j, \tag{21}$$

and b_r is the appropriate coefficient in (1.1), i.e., $10 + 2(r-1) \leqslant x_j \leqslant 10 + 2r$, $r = 1, 2, \ldots, 5$.

When the estimation model employed was the BCC model of DEA in (9), we determined $\hat{\gamma}_j$ in the following two-step manner. For observations identified as being on the frontier, we calculated the slope for each piecewise segment. For frontier observations on multiple segments of the frontier, we set $\hat{\gamma}_j$ as the average of the slopes of those segments. For each inefficient observation, we identified the reference segment on the frontier which was used to evaluate that observation and used the slope of this reference segment to estimate $\hat{\gamma}_j$. However, when the estimation models employed were the regression fitted DEA, COLS, MLE or A&C instead of DEA in (9), $\hat{\gamma}_j$ was estimated as the slope calculated from (10), (11), (12), (13) and (17) or (18) as follows:

$$\hat{\gamma}_j = \begin{cases} \hat{\beta}_1 \hat{g}(x_j)/x_j \text{ if the production function estimated is log-linear, and} \\ \left[(\hat{\beta}_1 + 2\hat{\beta}_{11} \ln x_j)\hat{g}(x_j)\right]/x_j \text{ if the production function estimated is log-quadratic.} \end{cases} \tag{22}$$

5. Results

We report our simulation results in Tables 1–6 which, for convenience of reference, are arranged in the same numerical order as the six cases discussed in Section 2.4 above. Each table has two panels. Panel A contains the mean and median absolute deviations of the efficient output from its true value and Panel B presents the mean and median absolute deviations of slope parameter. Each panel has 11 columns in which the first three columns present the results of DEA-based estimators, the six columns in the middle present the results of COLS-R, COLS-M and MLE estimators and the last two columns present the results of A&C estimators.

5.1. Impact of heteroscedasticity on DEA-based estimators

We begin by discussing the impact of heteroscedasticity on DEA-based estimators. The simulation results reported in Table 1 are for Case 1 when there is only a one-sided homoscedastic error term. From the first column and the first row of Panel A of Table 1, we observe that the average absolute deviation of the BCC estimator of the efficient output from its true value is 2.06. This average is taken over all 100 samples of size $N = 25$. The median value of 1.54 reported in row 2 was similarly calculated as the median of median absolute deviations over all 100 samples. The mean and median values for the other two sample sizes were calculated in a similar way to those for $N = 25$. In addition, in the final two rows, we also report the overall mean and median values across the 3 different sample sizes for each column. Since the one-sided error specification in this case is consistent with the postulates underlying the DEA estimator (Banker, 1993), we use this as a benchmark to evaluate the impact of heteroscedasticity on DEA-based estimators in the other cases considered in this study.

Turning to Table 2, we see that average absolute deviations of the BCC estimator of the efficient output and the slope parameter for each of the 3 different sample sizes in the presence of heteroscedasticity are close to their corresponding values reported in Table 1 when there is no heteroscedasticity. The overall

Table 1
Simulation results with only a one-sided error term and no heteroscedasticity

		DEA			COLS-R		COLS-M		MLE		A&C	
		BCC	Fitted LL	Fitted LQ	LL	LQ	LL	LQ	LL	LQ	LL	LQ
Panel A: Absolute deviations (AD) of output (100 iterations)												
$N = 25$	Mean of mean AD	2.06	2.29	2.16	1.97	3.03	10.94	10.91	8.99	9.08	34.98	29.76
	Median of median AD	1.54	2.30	1.66	1.03	1.82	11.44	10.93	9.86	9.32	36.81	29.95
$N = 50$	Mean of mean AD	1.31	1.69	1.38	1.49	2.32	10.70	10.69	8.94	8.96	40.22	35.46
	Median of median AD	0.97	1.70	1.04	0.76	1.38	11.07	10.79	9.94	9.28	40.55	35.12
$N = 100$	Mean of mean AD	0.85	1.31	0.89	1.32	1.81	10.87	10.87	9.43	9.33	43.73	41.41
	Median of median AD	0.60	1.34	0.69	0.63	1.19	11.28	10.81	10.18	9.38	44.13	41.76
Overall mean		1.40	1.76	1.47	1.59	2.38	10.83	10.82	9.12	9.12	39.64	35.54
Overall median		0.97	1.70	1.04	1.03	1.82	11.28	10.81	9.94	9.32	40.55	35.12
Panel B: AD of slope (100 iterations)												
$N = 25$	Mean of mean AD	0.84	0.82	0.88	0.95	1.46	0.95	1.27	0.94	1.25	1.67	1.78
	Median of median AD	0.54	0.73	0.65	0.76	1.09	0.81	0.99	0.76	0.98	1.43	1.44
$N = 50$	Mean of mean AD	0.71	0.73	0.57	0.81	0.95	0.85	0.89	0.83	0.87	1.35	1.98
	Median of median AD	0.47	0.69	0.43	0.76	0.77	0.75	0.82	0.73	0.79	1.28	1.54
$N = 100$	Mean of mean AD	0.57	0.71	0.44	0.76	0.71	0.81	0.70	0.80	0.68	1.29	2.21
	Median of median AD	0.34	0.71	0.33	0.72	0.52	0.75	0.65	0.75	0.60	1.17	2.10
Overall mean		0.70	0.75	0.63	0.84	1.04	0.87	0.96	0.85	0.93	1.43	1.99
Overall median		0.47	0.71	0.43	0.76	0.77	0.75	0.82	0.75	0.79	1.28	1.54

mean and median values reported in Table 2 for the BCC estimator are also similar to those reported in Table 1, confirming our expectation that heteroscedasticity does not impact the performance of the BCC efficiency estimator. Furthermore, the DEA estimator exhibits the desired statistical property of consistency proved in Banker (1993) with the average absolute deviations decreasing with sample size whether heteroscedasticity is present or not.

Table 3 reports the results for the case when both inefficiency and measurement error terms are present, and neither is heteroscedastic. The BCC model's performance in terms of the absolute deviations of its efficient output and slope estimators from their true values is substantially worse than the performance reported in Table 1. This is expected because the composed error specification in Case 3 violates the maintained assumptions of the DEA estimator. However, the performance of the BCC estimators reported in Tables 4–6 is not very different from their performance reported in Table 3, indicating that heteroscedasticity does not affect the performance of the DEA estimators even in the presence of measurement error. Observe also that in the latter four tables for the case when measurement error is present, the performance of the DEA estimators actually worsens with increased sample size as the distortive impact of measurement errors is enhanced. Therefore, comparing the group of results reported in Tables 1 and 2 with those reported in Tables 3–6, we observe that it is the presence of measurement error that affects the performance of the BCC estimators, but the presence of heteroscedasticity has no marked impact.

The results for the two-stage DEA with fitted parametric regression models are reported in the second and the third columns of Tables 1–6. Similar to the performance of the BCC model reported in the first column of those tables, the performance of the regression-fitted DEA models also deteriorated when the

Table 2
Simulation results with only a one-sided error term and heteroscedasticity in inefficiency

		DEA			COLS-R		COLS-M		MLE		A&C	
		BCC	Fitted LL	Fitted LQ	LL	LQ	LL	LQ	LL	LQ	LL	LQ
Panel A: AD of output (100 iterations)												
$N = 25$	Mean of mean AD	2.14	2.39	2.18	2.21	3.34	10.95	10.92	9.05	9.11	38.95	32.79
	Median of median AD	1.51	2.34	1.62	1.15	2.02	11.82	10.68	10.04	9.43	39.48	32.80
$N = 50$	Mean of mean AD	1.27	1.58	1.30	1.88	2.85	10.77	10.76	8.98	9.01	41.99	34.76
	Median of median AD	0.98	1.61	1.03	0.87	1.58	11.50	10.73	10.11	9.05	42.34	34.20
$N = 100$	Mean of mean AD	0.83	1.23	0.84	1.83	2.88	10.90	10.88	9.57	9.52	46.55	39.74
	Median of median AD	0.59	1.25	0.65	0.70	2.09	11.76	10.75	10.74	9.50	46.96	40.05
Overall mean		1.41	1.70	1.44	1.45	3.02	10.87	10.85	9.20	9.21	42.50	35.76
Overall median		0.98	1.61	1.03	0.87	2.02	11.76	10.73	10.11	9.43	42.34	34.20
Panel B: AD of slope (100 iterations)												
$N = 25$	Mean of mean AD	0.76	0.77	0.73	1.04	1.53	1.21	1.51	1.17	1.47	1.49	2.39
	Median of median AD	0.52	0.74	0.60	0.79	1.18	0.98	1.39	0.88	1.31	1.29	2.55
$N = 50$	Mean of mean AD	0.68	0.71	0.47	0.91	1.07	1.12	1.22	1.07	1.18	1.30	2.33
	Median of median AD	0.44	0.68	0.37	0.76	0.98	0.90	1.12	0.88	1.09	1.12	2.41
$N = 100$	Mean of mean AD	0.56	0.69	0.36	0.86	0.90	1.10	1.11	1.07	1.08	1.28	2.57
	Median of median AD	0.34	0.66	0.27	0.72	0.83	0.93	1.03	0.89	1.02	1.10	2.81
Overall mean		0.66	0.72	0.52	0.93	1.16	1.14	1.28	1.10	1.24	1.35	2.43
Overall median		0.44	0.68	0.37	0.76	0.98	0.93	1.12	0.88	1.09	1.12	2.56

measurement error term was introduced, but heteroscedasticity did not have any material impact on the performance of these DEA-based estimators.

5.2. Impact of heteroscedasticity on COLS, ML and A&C estimators

Next, we consider the impact of heteroscedasticity on the parametric frontier estimators. From columns 2 and 3 of Table 1, we observe that the average absolute deviation of the efficient output estimator using the LL and LQ forms of COLS-R from their true value is 1.97 and 3.03, respectively, when the sample size is 25 and a one-sided homoscedastic error term is specified. The corresponding values reported in Table 2 are 2.21 and 3.34 when the one-sided inefficiency term is specified to be heteroscedastic, suggesting that heteroscedasticity has only a small impact on the COLS-R estimators. The results for other sample sizes are also similar to those for $N = 25$. However, the performance of the COLS-R estimators reported earlier in both Tables 1 and 2 when only the inefficiency term is specified was much better than that in Table 3 when a measurement error is introduced. For instance, average absolute deviations of the efficient output estimator from the true value in both Tables 1 and 2 were only about one-third of those reported in Table 3. This result is consistent with that in Banker et al. (1993). In addition, since the performance of the COLS-R estimators reported in Tables 4–6 is not much different from that in Table 3, we surmise that the impact of heteroscedasticity on the COLS-R estimators is not as strong as that of measurement error.

Turning to the performance of the COLS-M estimators, we see from Tables 1 and 2 that the performance of both LL and LQ forms of the COLS-M estimators does not change much when we introduce heteroscedasticity in the one-sided error term, but the performance of the COLS-M slope estimators de-

Table 3
Simulation results with composed error term and no heteroscedasticity

		DEA			COLS-R		COLS-M		MLE		A&C	
		BCC	Fitted LL	Fitted LQ	LL	LQ	LL	LQ	LL	LQ	LL	LQ
Panel A: AD of output (100 iterations)												
$N = 25$	Mean of mean AD	3.49	3.01	3.50	6.34	6.79	10.84	10.82	9.48	9.56	39.08	34.16
	Median of median AD	2.68	2.06	3.04	5.36	6.02	11.73	11.10	10.15	9.58	38.44	33.95
$N = 50$	Mean of mean AD	4.66	4.12	4.58	8.44	8.46	11.01	10.99	9.52	9.52	40.83	36.73
	Median of median AD	4.42	2.96	4.53	7.53	8.31	11.53	11.15	10.26	9.66	40.85	35.71
$N = 100$	Mean of mean AD	6.23	5.89	6.11	9.97	9.84	10.94	10.93	9.41	9.43	43.88	41.58
	Median of median AD	6.12	4.93	6.41	8.76	9.50	11.50	10.97	10.01	9.45	44.38	41.42
Overall mean		4.79	4.34	4.50	8.25	8.36	10.92	10.91	9.47	9.50	41.20	37.49
Overall median		4.42	2.96	4.53	7.53	8.31	11.53	11.10	10.15	9.58	40.85	35.71
Panel B: AD of slope (100 iterations)												
$N = 25$	Mean of mean AD	1.58	1.00	1.54	0.95	1.62	0.90	1.37	0.90	1.36	1.58	2.31
	Median of median AD	0.98	0.78	1.32	0.73	1.39	0.77	1.18	0.79	1.19	1.27	2.20
$N = 50$	Mean of mean AD	1.54	0.97	1.50	0.93	1.27	0.88	1.05	0.87	1.05	1.34	1.82
	Median of median AD	1.04	0.73	1.32	0.71	1.00	0.76	0.91	0.76	0.93	1.22	1.30
$N = 100$	Mean of mean AD	1.47	0.92	1.34	0.86	1.02	0.80	0.80	0.78	0.79	1.22	2.23
	Median of median AD	1.05	0.76	1.14	0.71	0.75	0.74	0.66	0.77	0.65	1.11	2.28
Overall mean		1.53	0.96	1.46	0.91	1.30	0.86	1.07	0.85	1.06	1.38	2.12
Overall median		1.04	0.76	1.32	0.71	1.00	0.76	0.91	0.77	0.93	1.22	2.20

teriorates. Comparing the results reported in Tables 3–6 when a composed error term is specified, we observe a similar pattern to that in Tables 1 and 2. Heteroscedasticity does not seem to impact the COLS-M output estimators. Interestingly, the COLS-M slope estimator is adversely affected by heteroscedasticity in the inefficiency component but not affected by heteroscedasticity in the measurement component.

The performance of both ML and A&C estimators is reported in the final two columns of Tables 1–6. Once again, we observe very similar patterns. Performance of the output estimators does not seem to be affected by heteroscedasticity. Performance of the slope estimators is adversely affected by heteroscedasticity in the inefficiency term, but not affected by heteroscedasticity in the measurement error. In general, we observe that there is not much difference between the output estimators of the LL and LQ forms of any method. However among the slope estimators in the presence of measurement error, the LL form performs better than the LQ form for all methods because the more flexible LQ form is likely susceptible to being misled by those errors. It appears that there is a cost to flexibility when there is measurement error.

5.3. Comparative performance

Finally, we turn to the evaluation of the comparative performance of the five types of estimators. Looking at the last two rows of Panel A of Table 1, we see that overall mean and median absolute deviations of the efficient output estimator using the BCC model are 1.40 and 0.97, respectively, indicating that the BCC estimator is the best performer in estimating efficient output. While both the regression-fitted DEA and COLS-R estimators perform equally well in terms of overall mean and median absolute deviations of the efficient output, both estimators outperform all COLS-M, MLE and A&C estimators. In fact,

Table 4
Simulation results with composed error term and heteroscedasticity in inefficiency

		DEA			COLS-R		COLS-M		MLE		A&C	
		BCC	Fitted LL	Fitted LQ	LL	LQ	LL	LQ	LL	LQ	LL	LQ
Panel A: AD of output (100 iterations)												
$N = 25$	Mean of mean AD	3.40	2.83	3.33	6.48	7.17	10.87	10.84	9.55	9.64	37.10	32.42
	Median of median AD	2.58	1.79	3.06	5.43	6.27	11.47	10.71	9.96	9.30	36.40	31.44
$N = 50$	Mean of mean AD	4.63	4.15	4.53	8.50	8.83	11.04	11.01	9.62	9.62	43.61	36.12
	Median of median AD	4.32	3.14	4.73	7.57	8.56	12.02	10.92	10.62	9.37	43.50	35.92
$N = 100$	Mean of mean AD	6.22	5.96	6.11	10.07	10.08	10.96	10.95	9.51	9.51	47.51	41.62
	Median of median AD	6.18	5.13	6.52	9.17	10.15	11.97	10.85	10.48	9.37	49.06	42.75
Overall mean		4.75	4.31	4.65	8.35	8.69	10.95	10.93	9.56	9.59	42.66	36.72
Overall median		4.32	3.14	4.73	7.57	8.56	11.97	10.85	10.48	9.37	43.50	35.92
Panel B: AD of slope (100 iterations)												
$N = 25$	Mean of mean AD	1.47	0.91	1.40	0.93	1.61	1.16	1.54	1.13	1.53	2.03	2.75
	Median of median AD	1.07	0.76	1.23	0.74	1.38	0.97	1.50	0.93	1.49	1.43	3.28
$N = 50$	Mean of mean AD	1.52	0.88	1.37	0.87	1.28	1.14	1.33	1.11	1.31	1.27	2.41
	Median of median AD	1.01	0.73	1.23	0.73	1.13	0.96	1.23	0.91	1.22	1.18	2.52
$N = 100$	Mean of mean AD	1.49	0.85	1.25	0.75	0.92	1.07	1.15	1.04	1.11	1.20	2.45
	Median of median AD	1.03	0.72	1.13	0.65	0.77	0.99	1.09	0.94	1.06	1.10	2.61
Overall mean		1.49	0.88	1.34	0.85	1.27	1.12	1.34	1.09	1.31	1.50	2.53
Overall median		1.03	0.73	1.23	0.73	1.13	0.97	1.23	0.93	1.22	1.18	2.61

the A&C estimators of efficient output perform the worst among the five types of frontier estimators we considered. The results reported in Table 2 are very similar to those reported in Table 2, indicating that DEA-based estimators are the best estimators of efficient output even under heteroscedasticity when only one-sided error is present.

To get a better picture of the comparative performance among different estimators, we simplify the presentations of these results by ranking them in Tables 7 and 8. In each of these tables, the numerical values correspond to rankings for the overall mean and overall median absolute deviation values reported in the preceding six tables. These rankings represent the relative superiority of the approaches being examined according to whether one approach outperforms the other in terms of overall mean and overall median values reported in the final two rows of the Tables 1–6. [4]

Turning to Table 8 we find once again that the three DEA-based slope estimators perform the best when there is no measurement error present. The last four rows of Table 8, however, indicate that the LL form of COLS-R slope estimator performs the best under measurement error, with the LL forms of MLE and regression-fitted DEA close behind.

As indicated by the ranks in Table 7, the three DEA-based output estimators outperform all of the other approaches. The BCC model of DEA, as might be expected, has the lowest mean and the lowest median in the first two cases when there is no measurement error. Furthermore, inspection of the last 4 rows of Table

[4] Ties, when they occurred, were resolved by reporting the average of the tied rank scores for all tied methods in order to preserve the full (1–11) range of the numerical orders that are reported in Tables 7 and 8, respectively.

Table 5
Simulation results with composed error term and heteroscedasticity in measurement error

		DEA			COLS-R		COLS- M		MLE		A&C	
		BCC	Fitted LL	Fitted LQ	LL	LQ	LL	LQ	LL	LQ	LL	LQ
Panel A: AD of output (100 iterations)												
$N = 25$	Mean of mean AD	3.69	3.37	3.74	6.39	6.92	10.85	10.83	9.42	9.55	37.52	33.50
	Median of median AD	2.72	2.33	3.17	5.06	5.99	11.61	11.12	10.19	9.66	38.15	34.84
$N = 50$	Mean of mean AD	4.88	4.42	4.86	8.45	8.66	11.02	11.00	9.53	9.56	40.68	36.43
	Median of median AD	4.32	3.07	4.53	7.27	7.82	11.62	11.05	10.24	9.67	41.32	37.13
$N = 100$	Mean of mean AD	6.48	6.15	6.40	10.25	10.30	10.91	10.90	9.39	9.37	46.08	41.99
	Median of median AD	6.31	4.82	6.67	8.96	10.25	11.47	11.03	9.97	9.40	46.78	43.86
Overall mean		5.01	4.64	5.00	8.36	8.62	10.92	10.91	9.44	9.49	41.40	37.30
Overall median		4.22	3.07	4.53	7.27	7.82	11.61	11.05	10.19	9.66	41.32	37.13
Panel B: AD of slope (100 iterations)												
$N = 25$	Mean of mean AD	1.62	1.15	1.53	0.95	1.64	0.91	1.37	0.90	1.38	1.60	2.05
	Median of median AD	1.08	0.87	1.22	0.76	1.39	0.77	1.23	0.79	1.23	1.28	1.75
$N = 50$	Mean of mean AD	1.71	1.17	1.52	0.92	1.27	0.88	1.05	0.87	1.05	1.33	1.84
	Median of median AD	1.16	0.89	1.31	0.70	1.03	0.75	0.99	0.74	0.95	1.23	1.44
$N = 100$	Mean of mean AD	1.74	1.19	1.43	0.86	1.01	0.80	0.79	0.78	0.77	1.26	2.15
	Median of median AD	1.29	1.04	1.26	0.72	0.80	0.74	0.69	0.76	0.66	1.14	1.66
Overall mean		1.69	1.17	1.49	0.91	1.30	0.86	1.07	0.85	1.06	1.39	2.01
Overall median		1.16	0.89	1.26	0.72	1.03	0.75	0.99	0.76	0.95	1.23	1.66

7 shows that LL (= log linear fit) under DEA has top rank in all four cases when measurement error was present. Heteroscedasticity, does *not* alter these relative rankings.

5.4. Sensitivity analysis

To evaluate the sensitivity of the simulation results to our earlier piecewise Cobb–Douglas specification of the production technology, we also estimated a single smooth Cobb–Douglas type production function. Specifically, to introduce both increasing and decreasing returns to scale [5] in a parsimonious manner, we used the following "shifted" Cobb–Douglas production function:

$$y^* = g(x) = \alpha(x - 5)^{\beta}, \tag{23}$$

[5] For example, consider scale elasticity defined by

$$e(x) = \frac{\partial g(\lambda x)}{\partial \lambda}\frac{\lambda}{g(x)}\bigg|_{\lambda=1}$$

with $e(x) \gtreqless 1$ according to whether increasing, constant or decreasing returns prevail at x (Varian, 1992, p. 17). Thus, for (23) with $\alpha = 10$, $\beta = 0.6$, we have

$$e(x) = \frac{6x}{10(x-5)}, \quad 10 \leqslant x \leqslant 20.$$

For $x = 10$ this gives $e(x) = (60/50) > 1$ for increasing returns to scale. For $x = 20$ we get $e(x) = (120/150) < 1$, for decreasing returns. Finally for $x = 12.5$ we get $e(x) = 1$ for constant returns. Hence, all of these three different returns to scale are covered, as claimed.

Table 6
Simulation results with composed error term and heteroscedasticity in both inefficiency and measurement error

		DEA			COLS-R		COLS-M		MLE		A&C	
		BCC	Fitted LL	Fitted LQ	LL	LQ	LL	LQ	LL	LQ	LL	LQ
Panel A: AD of output (100 iterations)												
$N = 25$	Mean of mean AD	3.43	3.00	3.39	6.74	7.48	10.87	10.84	9.64	9.69	27.86	23.77
	Median of median AD	2.64	1.77	2.95	5.27	6.40	11.53	10.70	10.01	9.32	25.38	20.16
$N = 50$	Mean of mean AD	4.77	4.38	4.71	9.21	9.67	11.10	11.07	9.68	9.69	29.98	26.59
	Median of median AD	4.40	3.15	4.58	7.85	8.91	12.04	10.95	10.66	9.53	26.81	22.78
$N = 100$	Mean of mean AD	6.37	6.14	6.29	11.07	11.44	10.92	10.91	9.46	9.44	31.42	29.91
	Median of median AD	6.18	4.88	6.54	9.79	11.34	11.87	10.82	10.37	9.22	27.83	23.70
Overall mean		4.85	4.50	4.79	9.00	9.53	10.96	10.94	9.59	9.60	29.75	26.75
Overall median		4.40	3.15	4.58	7.85	8.91	11.87	10.82	10.37	9.32	26.81	22.78
Panel B: AD of slope (100 iterations)												
$N = 25$	Mean of mean AD	1.52	0.99	1.34	0.94	1.59	1.16	1.53	1.14	1.53	2.40	1.91
	Median of median AD	1.08	0.79	1.10	0.74	1.32	0.97	1.48	0.94	1.45	2.53	1.70
$N = 50$	Mean of mean AD	1.60	1.05	1.36	0.87	1.29	1.15	1.33	1.12	1.31	2.06	1.87
	Median of median AD	1.13	0.79	1.20	0.74	1.16	1.00	1.22	0.95	1.23	2.11	1.74
$N = 100$	Mean of mean AD	1.69	1.08	1.31	0.74	0.92	1.06	1.14	1.03	1.11	1.83	1.87
	Median of median AD	1.32	0.89	1.14	0.67	0.78	0.95	1.10	0.90	1.06	1.86	1.53
Overall mean		1.60	1.04	1.33	0.85	1.26	1.12	1.33	1.09	1.31	2.09	1.88
Overall median		1.13	0.79	1.14	0.74	1.16	0.97	1.22	0.94	1.23	2.11	1.70

Table 7
Efficient output estimator performance

Tables	DEA			COLS-R		COLS-M		MLE		A&C	
	BCC	Fitted LL	Fitted LQ	LL	LQ	LL	LQ	LL	LQ	LL	LQ
Panel A: Rankings based on mean absolute deviations											
1	1	4	2	3	5	9	8	6.5	6.5	11	10
2	1	4	2	3	5	9	8	6	7	11	10
3	3	1	2	4	5	9	8	6	7	11	10
4	3	1	2	4	5	9	8	6	7	11	10
5	3	1	2	4	5	9	8	6	7	11	10
6	3	1	2	4	5	9	8	6	7	11	10
Panel B: Rankings based on median absolute deviations											
1	1	4	3	2	5	9	8	7	6	11	10
2	2	4	3	1	5	9	8	7	6	11	10
3	2	1	3	4	5	9	8	7	6	11	10
4	2	1	3	4	5	9	8	7	6	11	10
5	2	1	3	4	5	9	8	7	6	11	10
6	2	1	3	4	5	9	8	7	6	11	10

where $\alpha = 10$, $\beta = 0.6$ and the input x was generated randomly from a uniform probability distribution over the interval [10, 20].

Table 8
Slope estimator performance

Tables	DEA			COLS-R		COLS-M		MLE		A&C	
	BCC	Fitted LL	Fitted LQ	LL	LQ	LL	LQ	LL	LQ	LL	LQ
Panel A: Rankings based on mean absolute deviations											
1	2	3	1	4	9	6	8	5	7	10	11
2	2	3	1	4	7	6	9	5	8	10	11
3	10	4	9	3	7	2	6	1	5	8	11
4	9	2	7.5	1	5	4	7.5	3	6	10	11
5	10	6	9	3	7	2	5	1	4	8	11
6	9	2	7.5	1	5	4	7.5	3	6	11	10
Panel B: Rankings based on median absolute deviations											
1	2	3	1	6	7	4.5	9	4.5	8	10	11
2	2	3	1	4	7	6	9.5	5	8	9.5	11
3	8	3.5	10	1	7	2	5	3.5	6	9	11
4	10	1.5	8.5	1.5	5	4	8.5	3	7	6	11
5	8	4	10	1	7	2	6	3	5	9	11
6	5	2	6	1	7	4	8	3	9	11	10

We repeated the estimation procedures for the 6 cases discussed above. The simulation results (not reported here) are similar to those discussed earlier when a piecewise Cobb–Douglas production function was specified, indicating that heteroscedasticity does not adversely impact DEA-based estimators. While the relative performance patterns are similar to those discussed earlier, the performance of the A&C-LL estimator in estimating the slope parameter improves substantially over that reported in the earlier experiment.

Collectively, our results indicate that heteroscedasticity does not materially impact the relative performance of the different efficiency estimation methods. The BCC model for DEA performs the best regardless of the presence of heteroscedasticity when there is no measurement error. In the presence of measurement error, the two-stage DEA followed by a fitted log-linear form for the output estimator generally performs the best regardless of the presence of heteroscedasticity.

6. Conclusion

This paper reports a Monte Carlo simulation study to evaluate the impact of heteroscedasticity on DEA and other frontier estimators, and compare the relative performance of DEA, COLS-R, COLS-M, MLE and A&C estimators in the presence of heteroscedasticity. The results indicate that heteroscedasticity does not adversely affect the performance of DEA-based estimators. In fact, the impact of the nature of the error term (one-sided error versus composed error) is much greater than that of heteroscedasticity. DEA proved to be the best estimator of efficient output even when errors were heteroscedastic. This is consistent with other studies although we have here extended these previous studies to comprehend the case of heteroscedasticity and the uses of DEA and DEA-regression combinations to estimate slope parameters. Particularly striking was the performance of the DEA-log-linear regression combination which in all four cases when measurement error was present, bested the performance of the BCC model of DEA both as an efficient output estimator as well as slope estimator.

This is about as far as our use of Tables 7 and 8 will carry us in our evaluation of relative performances. We have here used very simple (both non-linear and linear) models. Hence the results we report should be regarded as a beginning rather than an end to studies of the effects of heteroscedasticity. Further extensions could include more general models as well as other methods of estimation. Like others who have conducted such simulation studies, we have focused on "closeness of fit" as the criterion of performance. However, other statistical properties, such as "robustness" can also be important. Finally, over and beyond the present study, we might note that extensions to examine the effects of misspecification as well as other possible types of problems could also be profitably undertaken along lines like those initiated in Banker et al. (1996).

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