Econometrica, Vol. 56, No. 6 (November, 1988), 1315–1332

NONPARAMETRIC ANALYSIS OF TECHNICAL AND ALLOCATIVE EFFICIENCIES IN PRODUCTION

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In this paper we extend Varian's (1984) nonparametric production analysis to situations when the set of observed output, input, and price data is not consistent with profit maximization for at least one firm. In such cases, Varian's results imply that no production possibility set containing all observations can *rationalize* the observed data. We identify each firm whose performance, given the prices faced by it, may be found consistent with profit maximization relative to *some* production possibility set containing all observed output-input vectors. We show that the set $\mathscr{E}$ of *all* such firms can itself be *weakly* rationalized in the sense that there exists a (closed, convex, and "monotone") production possibility set that contains all the observations, and relative to which the performance of all the firms in the set $\mathscr{E}$ is consistent with profit maximization given their respective prices.

By definition, firms not included in this largest set $\mathscr{E}$ of *efficient* observations unambiguously deviate from profit maximizing behavior for *any* production possibility set containing all observations. We follow Farrell (1957) and analyze these deviations into technical and allocative efficiency measures, considering as *admissible* all closed, convex, and "monotone" production possibility sets relative to which the performance of each firm in the set $\mathscr{E}$ remains consistent with profit maximization. We then describe nonparametric methods for determining the *tightest* upper and lower bounds on the technical, allocative, and aggregate efficiency measures evaluated relative to all such admissible production possibility sets. It is seen that the tightest upper bound on the technical efficiency measure is the same as the value computed by the nonparametric efficiency evaluation technique known as data envelopment analysis, thus establishing a link between this literature in management science/operations research and the nonparametric production analysis in economics.

KEYWORDS: Nonparametric estimation, production analysis, profit maximization, rationalization, data envelopment analysis, technical efficiency, allocative efficiency.

1. INTRODUCTION

THE CONVENTIONAL APPROACH to the estimation of production functions consists of first specifying a parametric form for the function and then fitting it to observed data by minimizing some measure of their distance from the estimated function. Statistical tests are performed by postulating again a parametric form for the distribution of the deviations of observed data from the fitted production function. The fundamental weakness of this approach lies in its inability to theoretically substantiate or statistically test the maintained hypotheses about the parametric form for the production function and the postulated distribution for the disturbance term. Furthermore, it is not immediately apparent what restrictions these hypotheses, treated as axioms in the conventional approach, impose on the production correspondence.

In an alternative nonparametric approach to production analysis, Varian (1984) shows that the consistency of the observed input, output, and price data with profit maximization is equivalent to the existence of an underlying family of closed and convex production possibility sets, each of which *rationalizes* the

[1] Helpful comments and suggestions by three anonymous referees are gratefully acknowledged.

observed data.[2] A production possibility set T is said to rationalize observed input, output, and price data for a set $\mathscr{I}$ of n firms, if each observed output-input vector is contained in the set T, and is also consistent with profit maximization relative to all the other production possibilities in T, for the corresponding observed price vector. Thus when the data set is consistent with profit maximization by each firm in the sample, Varian shows how one may bound the underlying technology without assuming a parametric functional form.

In many cases, however, the observed data will not be consistent with Varian's profit maximization postulate because of technical or allocative inefficiencies existing in the operations of some of the firms. Varian's results then also imply that there exists *no* production possibility set that can rationalize the observed data. In this paper we extend Varian's approach by referring to his definition as *strong rationalization* and introducing the notions of *subset rationalization* and *weak rationalization*. A production possibility set T is said to subset rationalize the set of observations $\mathscr{H} \subseteq \mathscr{I}$ if the output-input vector for *all* the observations $i \in \mathscr{I}$ are contained in T and every observation $i \in \mathscr{H}$ is consistent with profit maximization relative to all the production possibilities in T. We establish that there exists[3] a unique subset rationalizable set $\mathscr{E} \subseteq \mathscr{I}$ that includes every subset rationalizable subset of $\mathscr{I}$. A production possibility set T that subset rationalizes this largest subset rationalizable set $\mathscr{E}$ is then said to weakly rationalize $\mathscr{I}$. We also show that it is always possible to construct a closed, convex, and *monotone*[4] production possibility set that weakly rationalizes $\mathscr{I}$. In particular, the unique production possibility set[5] estimated via the axiomatic nonparametric approach of Banker (1980) and Banker, Charnes, and Cooper (1984), known as data envelopment analysis (DEA),[6] is shown to be one such set.

In general, many closed and convex sets may weakly rationalize $\mathscr{I}$. We consider as *admissible* those closed, convex, and *monotone* production possibility sets that weakly rationalize $\mathscr{I}$. Without assuming a specific functional form, only bounds can be placed on the production technology. We determine the *tightest* outer and inner bounds for the technology as in Varian. It is particularly interesting to note that the inner bound set is the same as the DEA estimate, thus providing a desired link[7] between two previously separate literatures.

[2] Varian's work draws upon earlier work by Samuelson (1947), Afriat (1972), Hanoch and Rothschild (1972), and Diewert and Parkan (1984).

[3] Note that it is, of course, possible that $\mathscr{E}$ is an empty set.

[4] As in Varian (1984), we refer to production possibility sets satisfying the "free disposal" property as "monotone."

[5] When the observed data set is consistent with profit maximization then, of course, $\mathscr{E} = \mathscr{I}$ and this production possibility set strongly rationalizes $\mathscr{I}$.

[6] DEA is a linear programming method developed for efficiency evaluation. There is considerable work in management science/operations research beginning with Charnes, Cooper, and Rhodes (1978, 1981). Banker (1984) and Banker, Charnes, and Cooper (1984) examine estimation of increasing/decreasing returns to scale, Banker, Charnes, Cooper, and Schinnar (1981) and Banker and Maindiratta (1986) consider increasing marginal products situations, and Banker and Morey (1986a, b) focus on situations when some inputs/outputs are fixed or when they are measured on a categorical scale. A comparison of DEA and parametric production analysis is provided by Banker, Conrad, and Strauss (1986).

[7] See, for instance, Lovell and Schmidt (1982, p. 14).

Relative to any *admissible* production possibility set, technical and allocative efficiencies can be analyzed in a manner analogous to Farrell (1957). By definition, all observations in $\mathscr{E}$ will be technically and allocatively efficient with respect to *all* admissible production possibility sets. Further, observations that are not in $\mathscr{E}$ are not consistent with profit maximization for *any* production possibility set containing $\mathscr{I}$. These observations are thus unambiguously technically and/or allocatively inefficient. The measures of technical, allocative, and aggregate (= technical × allocative) efficiencies will depend on the particular admissible production possibility set relative to which they are evaluated. In this paper, we also provide nonparametric methods for computing the *tightest* lower and upper bounds for each of these measures. These bounds represent the best that can be accomplished without imposing additional structure, such as by assuming a specific functional form.

2. PROFIT MAXIMIZATION AND RATIONALIZING OF OBSERVED DATA

We consider situations where we have observed data on the output-input vectors $(y^i, -x^i)$ and price vectors (p^i, w^i) for each firm $i \in \mathscr{I}$, where $\mathscr{I}$ is an index set of cardinality n. Here $y^i \geqslant 0$ and $p^i \geqslant 0$ are s-dimensional vectors of quantities and prices for s outputs, and $x^i > 0$[8] and $w^i \geqslant 0$ are m-dimensional vectors of quantities and prices for m inputs.

Varian (1984) introduced the notion of "rationalization," which we shall refer to as "strong rationalization" defined below:

DEFINITION 1: A production possibility set T is said to *strongly rationalize* the set of observations $\mathscr{I}$ if for each $i \in \mathscr{I}, (y^i, -x^i) \in T$ and $p^1 y^i - w^i x^i \geqslant p^i y - w^i x$ for all $(y, -x) \in T$.

However, due to technical and allocative inefficiencies in the operations of some of the firms, in many cases the set of observations may not be strongly rationalizable. Therefore, we shall next introduce the notion of subset rationalization.

DEFINITION 2: A production possibility set T is said to *subset rationalize* the set $\mathscr{H} \subseteq \mathscr{I}$ if for each $i \in \mathscr{I}$, $(y^i, -x^i) \in T$, and for each $i \in \mathscr{H}$, $p^i y^i - w^i x^i \geqslant p^i y - w^i x$ for all $(y, -x) \in T$. Further, a set of observations $\mathscr{H} \subseteq \mathscr{I}$ is said to be *subset rationalizable* if there exists some production possibility set T that can subset rationalize $\mathscr{H}$.

Next we turn to the problem of identifying observation sets that are subset rationalizable. For this purpose we define a criterion function Δ_i for each $i \in \mathscr{I}$:

$$\Delta_i = \max\left\{(p^i y^j - w^i x^j) - (p^i y^i - w^i x^i) \,\middle|\, j \in \mathscr{I}\right\}.$$

[8] The notation $x^i > 0$ indicates that each element of x^i is greater than zero.

Evidently, $\Delta_i \geqslant 0$ for all $i \in \mathcal{I}$. The inclusion of an observation $i \in \mathcal{I}$ in any subset rationalizable set $\mathcal{H} \subseteq \mathcal{I}$ will be possible only if $\Delta_i = 0$ as is evident from the following:

PROPOSITION 1: *The set* $\mathcal{E} \equiv \{i \mid \Delta_i = 0\} \subseteq \mathcal{I}$ *is subset rationalizable, and if* $\mathcal{H} \subseteq \mathcal{I}$ *is subset rationalizable then* $\mathcal{H} \subseteq \mathcal{E}$.[9]

Thus, if an observation $i \in \mathcal{I}$ is found to be consistent with profit maximization for *any* production possibility set T containing all the observations in $\mathcal{I}$, then such an observation $i \in \mathcal{E}$. Further, there exists at least one production possibility set containing all observations, for which every observation in this set $\mathcal{E}$ is found to be consistent with profit maximization.

The fact that there exists a unique subset rationalizable set $\mathcal{E} \subseteq \mathcal{I}$ containing every subset rationalizable set $\mathcal{H} \subseteq \mathcal{I}$, forms the basis of our definition of weak rationalization.

DEFINITION 3: A production possibility set T is said to *weakly rationalize* the observed data if it subset rationalizes $\mathcal{E} \equiv \{i \mid \Delta_i = 0\} \subseteq \mathcal{I}$.

If the observed data are not consistent with Varian's (1984) profit maximization postulate, then *no* production possibility set can *strongly* rationalize the observed data. However, we have the following important theorem.

PROPOSITION 2: *There always exists a closed and convex production possibility set, given by* $S \equiv \{(y, -x) \mid y \leqslant \Sigma_{i \in \mathcal{I}} \lambda_i y^i,\ x \geqslant \Sigma_{i \in \mathcal{I}} \lambda_i x^i,\ \Sigma_{i \in \mathcal{I}} \lambda_i = 1,\ y \geqslant 0,\ x \geqslant 0$ *and* $\lambda_i \geqslant 0\}$, *that weakly rationalizes the observed data.*

The set S described above can also be deduced[10] from the following set of four basic postulates provided by Banker, Charnes, and Cooper (1984) in the DEA literature:

POSTULATE 1—*Convexity:* If $(y^j, -x^j) \in S$, $j \in \mathcal{I}$, and $\lambda_j > 0$ with $\Sigma_{j \in \mathcal{I}} \lambda_j = 1$, then $(\Sigma_{j \in \mathcal{I}} \lambda_j y^j, -\Sigma_{j \in \mathcal{I}} \lambda_j x^j) \in S$.

POSTULATE 2—*Monotonicity:* If $(y, -x) \in S$ and $0 \leqslant y' \leqslant y$, $x' \geqslant x$, then $(y', -x') \in S$.

POSTULATE 3—*Inclusion of Observations:* All observed output-input vectors $(y^i, -x^i) \in S$ for all $i \in \mathcal{I}$.

[9] Proofs for all propositions are provided in the Appendix.

[10] See Banker (1987) for a detailed proof, where it is also shown that this method provides maximum likelihood estimates if the one-sided (nonnegative) radial inefficiency deviations ($\geqslant 1$) have a nonincreasing density function. Further, the DEA estimates are consistent if there is a positive probability that inefficiency deviations less than $1 + \delta$ will occur for any $\delta > 0$.

POSTULATE 4—*Minimum Extrapolation:* If T satisfies Postulates 1, 2, and 3, then $S \subseteq T$.

Convexity and monotonicity are desirable regularity properties commonly imposed on production possibility sets in production analysis. Caves and Christensen (1980, p. 431), for instance, suggest consistency with these properties as a criterion for choosing between different parametric forms. The inclusion of observations and the minimum extrapolation postulates ensure that the estimation of the production possibility set is based on the actual range of observed input and output data. These also imply that the set S will be the smallest convex set that includes all observations and satisfies the monotonicity property.

3. TECHNICAL, ALLOCATIVE, AND AGGREGATE EFFICIENCY MEASURES

In this section we integrate Varian's nonparametric production analysis with Farrell's analysis of technical and allocative efficiency when the set $\mathscr{E}$ of efficient observations is not empty. Toward this end we begin by specifying the class of admissible production possibility sets relative to which the efficiency measures may be computed.

DEFINITION 4: A production possibility set T belongs to the *class of admissible sets* $\mathscr{A}$ if and only if:

1(a) For each $i \in \mathscr{E}$, $p^i y^i - w^i x^i \geqslant p^i y - w^i x$, for all $(y, -x) \in T$.

1(b) For all observations $j \in \mathscr{I}$, the observed output-input vector $(y^j, -x^j) \in T$.

1(c) T is closed and convex.

1(d) If $(y, -x) \in T$ and $0 \leqslant y' \leqslant y$, $x' \geqslant x$, then $(y', -x') \in T$.

The first two conditions are equivalent to the condition that the production possibility set T weakly rationalizes the set of observed data $\mathscr{I}$. As a result, under our definitions of technical and allocative efficiency, to be introduced later, an actual observation will be evaluated as technically and allocatively efficient relative to *all* admissible production possibility sets if and only if it belongs to some subset $\mathscr{H} \subseteq \mathscr{I}$ that can be subset rationalized, or equivalently, by virtue of Proposition 1, this observation belongs to the efficient subset $\mathscr{E} \subseteq \mathscr{I}$. The efficiency of the observations in $\mathscr{E}$ cannot be refuted by the data, while the observations in $\mathscr{I}$ that are not in $\mathscr{E}$ are unambiguously inefficient. Therefore a criterion for admissibility of a production possibility set is that all observations in $\mathscr{E}$ remain efficient relative to it. The conditions of closure, convexity, and monotonicity for T are desirable regularity properties commonly imposed on production possibility sets in production analysis.

In addition, in many empirical applications, other restrictions on T such as minimum or maximum levels for particular inputs or outputs may be postulated. For instance, in order to bound the production possibility sets within the observed range of inputs and outputs, the minimum and the maximum observed values for each input and output may be specified as bounds on admissible production possibility sets. This will ensure that in our subsequent analysis of lower and upper bounds on the efficiency measures, the referent production possibilities for the efficiency evaluations are not projected beyond the observed range of observations. In the following developments we shall not explicitly specify such additional restrictions on the admissible sets. However, the results of Propositions 3, 4, 5, and 6 are not affected by any additional linear constraints imposing bounds on the levels of inputs and outputs for the admissible sets.

We shall now define technical, allocative, and aggregate efficiency measures relative to any production possibility set $T \in \mathscr{A}$ in a manner analogous to Farrell (1957).[11]

DEFINITION 5: The *aggregate efficiency* measure for an observation l with an actual profit $p^l y^l - w^l x^l > 0$, relative to the set $T \in \mathscr{A}$ is defined to be:

$$\gamma_T^l \equiv \min\left\{ (p^l y^l - w^l x^l) / (p^l y - w^l x) \middle| (y, -x) \in T, (p^l y - w^l x) \geqslant 0 \right\}.$$

DEFINITION 6: The *technical efficiency* measure for an observation l with an actual profit $p^l y^l - w^l x^l > 0$, relative to the set $T \in \mathscr{A}$ is defined to be:

$$\beta_T^l \equiv \min\left\{ (p^l y^l - w^l x^l) / (p^l y - w^l x) \middle| (y, -x) \in T, \right.$$
$$\left. y = y^l, x = hx^l, (p^l y - w^l x) \geqslant 0 \right\}.$$

Clearly, $0 < \gamma_T^l \leqslant 1$ and $0 < \beta_T^l \leqslant 1$ for all $T \in \mathscr{A}$ since $(y^l, -x^l) \in T$. Further, since any $(y^\beta, -x^\beta)$ solving the minimization problem for evaluating β_T^l necessarily belongs to T, it is feasible for the minimization problem for evaluating γ_T^l. Hence, $\gamma_T^l \leqslant \beta_T^l$.

DEFINITION 7: The *allocative efficiency* measure for an observation l with an actual profit $p^l y^l - w^l x^l > 0$, relative to the set $T \in \mathscr{A}$ is defined to be:

$$\alpha_T^l \equiv \gamma_T^l / \beta_T^l,$$

where γ_T^l and β_T^l are as defined above. Since $\gamma_T^l \leqslant \beta_T^l$, we have $0 < \alpha_T^l \leqslant 1$.

The above efficiency measures, and the associated referent points and supporting hyperplanes, will be different when determined relative to different admissible sets $T \in \mathscr{A}$. We next address the question of determining the individual lower and upper bounds on the technical, allocative, and aggregate efficiency measures of each observation relative to our class $\mathscr{A}$ of admissible production possibility sets. For this purpose, we have the following lemma and proposition:

[11] Note, however, that our measures are based on profit ratios, while Farrell's technical efficiency measure is radial, independent of prices. We also assume observed profit is positive, so that analysis using profit ratios is meaningful. At the end of this section we describe how our analysis extends to situations when this assumption is not valid.

LEMMA: *The two production possibility sets*:

$$L \equiv \{(y, -x) | p^i y - w^i x \leqslant p^i y^i - w^i x^i \ \forall i \in \mathscr{E},\ y \geqslant 0, x \geqslant 0\} \quad and$$

$$S \equiv \left\{(y, -x) | y \leqslant \sum_{i \in \mathscr{I}} \lambda_i y^i,\ x \geqslant \sum_{i \in \mathscr{I}} \lambda_i x^i,\ \sum_{i \in \mathscr{I}} \lambda_i = 1,\ y \geqslant 0, x \geqslant 0 \quad and \quad \lambda_i \geqslant 0\right\}^{12}$$

are admissible, that is L and $S \in \mathscr{A}$. Further, for any admissible production possibility set $T \in \mathscr{A}$, $S \subseteq T \subseteq L$.

Thus, in our extension of Varian's analysis these sets L and S correspond to his definition of the *outer bound* and the *inner bound*. It is particularly interesting to note that the inner bound set S corresponds to the DEA estimate, thus providing an important link between DEA and nonparametric production analysis in economics.

PROPOSITION 3: *The lower and upper bounds on the aggregate and technical efficiency measures of an observation $l \in \mathscr{I}$ evaluated over all admissible sets $T \in \mathscr{A}$ are given by*

(a) $\underline{\gamma}^l \equiv \min_{T \in \mathscr{A}} \gamma_T^l = \gamma_L^l$, (b) $\bar{\gamma}^l \equiv \max_{T \in \mathscr{A}} \gamma_T^l = \gamma_S^l$,

(c) $\underline{\beta}^l \equiv \min_{T \in \mathscr{A}} \beta_T^l = \beta_L^l$, (d) $\bar{\beta}^l \equiv \max_{T \in \mathscr{A}} \beta_T^l = \beta_S^l$.

Since the sets L and S are defined by linear inequalities, $\underline{\gamma}^l, \bar{\gamma}^l, \underline{\beta}^l, \bar{\beta}^l$ can all be determined by solving linear programs. For instance, $\underline{\gamma}^l = (p^l y^l - w^l x^l)/\pi_{\max}$ where: $\pi_{\max} = \max\{p^l y - w^l x | p^i y - w^i x \leqslant p^i y^i - w^i x^i$ for each $i \in \mathscr{E}$, $y, x \geqslant 0\}$. Further, $\bar{\gamma}^l = (p^l y^l - w^l x^l)/\pi_{\min}$, where $\pi_{\min} = \max\{p^l y - w^l x | y \leqslant \sum_{i \in \mathscr{I}} \lambda_i y^i$, $x \geqslant \sum_{i \in \mathscr{I}} \lambda_i x^i, 1 = \sum_{i \in \mathscr{I}} \lambda_i$, $y, x, \lambda_i \geqslant 0\}$. Since the feasible set for the latter linear program is a polytope whose corner points are some actual observations $j \in \mathscr{I}$, $\pi_{\min}$ can be computed simply by solving $\max\{p^l y^j - w^l x^j | j \in \mathscr{I}\}$. The lower and upper bounds on the technical efficiency measure can be computed by solving similar linear programs, with the additional linear constraints $y = y^l$, $x = hx^l$, and $h \geqslant 0$.

The lower and upper bounds on the allocative efficiency, that is $\underline{\alpha}^l = \min_{T \in \mathscr{A}} \alpha_T^l$ and $\bar{\alpha}^l \equiv \max_{T \in \mathscr{A}} \alpha_T^l$, cannot be determined as easily as the bounds on the aggregate and technical efficiencies, although evidently $\underline{\gamma}^l/\bar{\beta}^l \leqslant \underline{\alpha}^l \leqslant \bar{\alpha}^l \leqslant \bar{\gamma}^l/\underline{\beta}^l$. This additional complexity arises because $\alpha_T^l \equiv \gamma_T^l/\beta_T^l$ is evaluated as the ratio of the aggregate and technical efficiencies, and therefore, we have to simultaneously consider different possibilities for two referent points $(y^\gamma, -x^\gamma)$ and $(y^\beta, -x^\beta)$ to determine which production possibility set $T \in \mathscr{A}$ will minimize or maximize

[12] Note that S is the same production possibility set considered in Proposition 2.

the value of α_T^l. Hence, we first provide nonlinear programming formulations to characterize $\underline{\alpha}^l$ and $\bar{\alpha}^l$. We shall then use these characterizations to identify conditions under which these bounds can be easily computed, without resorting to the computational solution of the nonlinear program.

The two nonlinear programs have the same feasible space characterized by the following set of constraints:

2(a) $\quad p^l y^\gamma - w^l x^\gamma \geqslant p^l y^j - w^l x^j \quad \forall j \in \mathscr{I},$

2(b) $\quad p^i y^\gamma - w^i x^\gamma \leqslant p^i y^i - w^i x^i \quad \forall i \in \mathscr{E},$

2(c) $\quad p^\beta y^\beta - w^\beta x^\beta \geqslant p^\beta y^j - w^\beta x^j \quad \forall j \in \mathscr{I},$

2(d) $\quad p^i y^\beta - w^i x^\beta \leqslant p^i y^i - w^i x^i \quad \forall i \in \mathscr{E},$

2(e) $\quad y^\beta = y^l,$

2(f) $\quad x^\beta = h x^l,$

2(g) $\quad p^l y^\gamma - w^l x^\gamma \geqslant p^l y^\beta - w^l x^\beta,$

2(h) $\quad p^\beta y^\beta - w^\beta x^\beta \geqslant p^\beta y^\gamma - w^\beta x^\gamma,$

2(i) $\quad x^\gamma, y^\gamma, x^\beta, y^\beta, p^\beta, w^\beta \geqslant 0; \quad h \geqslant 0; \quad w^\beta \neq 0.$[13]

The variables in the constraint set are those listed in the nonnegativity constraints in 2(i). Note that the constraints in 2(c) and 2(h) are nonlinear.

PROPOSITION 4: *The upper bound $\bar{\alpha}^l$ and the lower bound $\underline{\alpha}^l$ on the allocative efficiency of observation $l \in \mathscr{I}$, relative to the admissible class of production possibility sets $\mathscr{A}$, are equal to $\bar{\psi}^l$ and $\underline{\psi}^l$ respectively, defined by the mathematical programs [P1] and [P2] described below*:

$$[P1]: \quad \bar{\psi}^l \equiv \max\left\{ (p^l y^\beta - w^l x^\beta)/(p^l y^\gamma - w^l x^\gamma) \middle| \textit{constraints } 2(a) \textit{ to } 2(i) \right\},$$

$$[P2]: \quad \underline{\psi}^l \equiv \min\left\{ (p^l y^\beta - w^l x^\beta)/(p^l y^\gamma - w^l x^\gamma) \middle| \textit{constraints } 2(a) \textit{ to } 2(i) \right\}.$$

In most cases, however, the determination of the lower and upper bounds on α^l does not require the actual computational solution of the nonlinear program. We have the following proposition for the upper bound $\bar{\alpha}^l$:

PROPOSITION 5: *A necessary and sufficient condition for $\bar{\alpha}^l$ to be equal to one is that the referent point $(y^{\underline{\beta}}, -x^{\underline{\beta}})$, solving the technical efficiency evaluation problem for observation l relative to the largest admissible set L, is also allocatively efficient for the prices (p^l, w^l) with respect to the observations $j \in \mathscr{I}$; that is, $p^l y^{\underline{\beta}} - w^l x^{\underline{\beta}} \geqslant p^l y^j - w^l x^j$ for all $j \in \mathscr{I}$. If $p^l y^{\underline{\beta}} - w^l x^{\underline{\beta}} < p^l y^j - w^l x^j$ for some $j \in \mathscr{I}$ then $\bar{\alpha}^l = \bar{\gamma}^l/\underline{\beta}^l$. Further, since $\bar{\alpha}^l \leqslant 1$ and also $\leqslant \bar{\gamma}^l/\underline{\beta}^l$, it follows that $\bar{\alpha}^l = \min\{1, \bar{\gamma}^l/\underline{\beta}^l\}$.*

Unfortunately, the computation of the lower bound for the allocative efficiency measure cannot be made as straightforward as the procedure for the upper bound

[13] The notation $w^\beta \neq 0$ indicates what w^β is not a null vector.

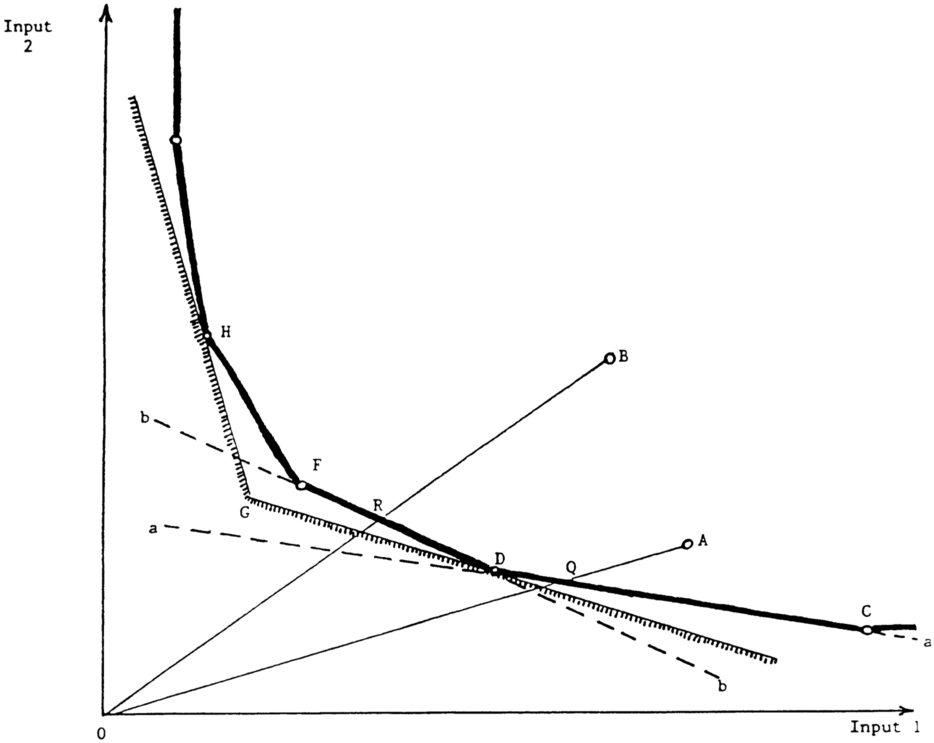

FIGURE 1.— ▬▬▬ Boundary of S, the *smallest* admissible set.
⊓⊓⊓⊓ Boundary of L, the *largest* admissible set.
The efficient subset of observations is $\mathscr{E} = \{D, H\}$.

$Q \equiv (y^{\bar{\beta}^A}, -x^{\bar{\beta}^A})$ is technically efficient referent point for A relative to the set S.
$R \equiv (y^{\bar{\beta}^B}, -x^{\bar{\beta}^B})$ is technically efficient referent point for B relative to the set S.
$G \equiv (y^{\underline{\gamma}^A}, -x^{\underline{\gamma}^A}) \equiv (y^{\underline{\gamma}^B}, -x^{\underline{\gamma}^B})$ is the referent point for evaluating the aggregate efficiency of both A and B relative to the set L.
$aDCa \equiv$ supporting hyperplane at Q for the set S.
$bFDb \equiv$ supporting hyperplane at R for the set S.

indicated above in Proposition 5. However, it is possible to identify a necessary and sufficient condition under which $\underline{\alpha}^l$ will be equal to $\underline{\gamma}^l/\bar{\beta}^l$. But, unlike the case of $\bar{\alpha}^l$, it is not possible to specify the value that $\underline{\alpha}^l$ will take when it is not equal to $\underline{\gamma}^l/\bar{\beta}^l$.

The necessary and sufficient condition specified in the next proposition requires checking if a hyperplane supporting the smallest admissible set S at the point $(y^{\bar{\beta}}, -x^{\bar{\beta}})$, solving the technical efficiency evaluation problem relative to S, also supports the convex closure of S and the point $(y^{\underline{\gamma}}, -x^{\underline{\gamma}})$ which solves the aggregate efficiency evaluation problem relative to the largest admissible set L. If (and only if) this condition holds then it is possible to construct an admissible production possibility set T relative to which the points $(y^{\underline{\gamma}}, -x^{\underline{\gamma}})$ and $(y^{\bar{\beta}}, -x^{\bar{\beta}})$ are the referents for evaluating the aggregate and technical efficiency respectively of observation l. Thus, in Figures 1 and 2 for instance, for observation A, $\underline{\alpha}^A = \underline{\gamma}^A/\bar{\beta}^A$, but for observation B, $\underline{\alpha}^B > \underline{\gamma}^B/\bar{\beta}^B$.

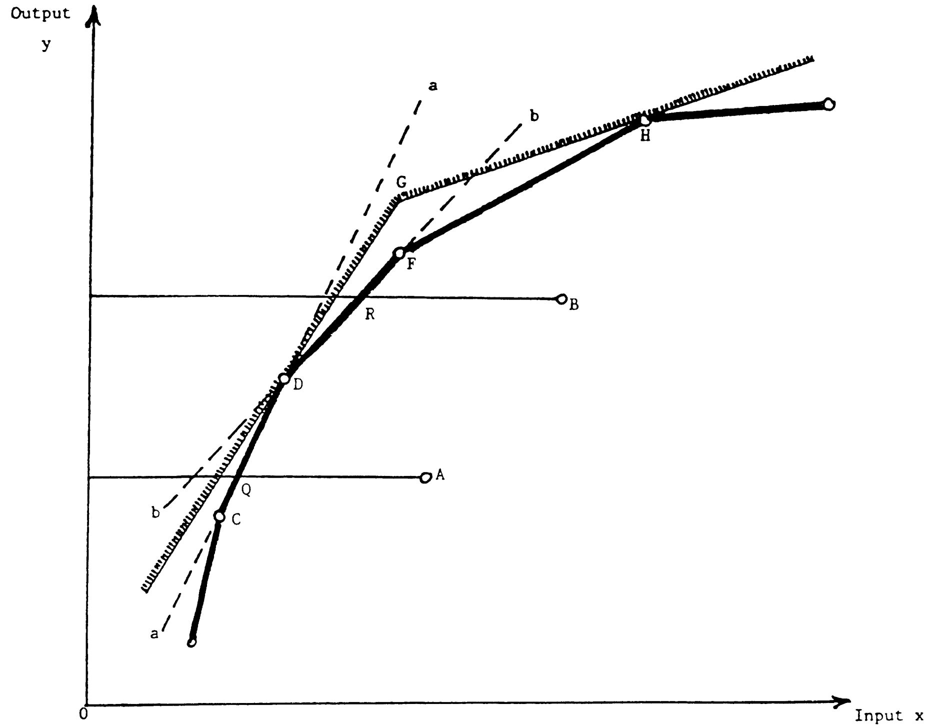

FIGURE 2.— ▬▬▬ Boundary of S, the *smallest* admissible set.
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$aDCa \equiv$ supporting hyperplane at Q for the set S.
$bFDb \equiv$ supporting hyperplane at R for the set S.

PROPOSITION 6: *The lower bound on the allocative efficiency measure $\underline{\alpha}^l$ for an observation $l \in \mathscr{I}$ is equal to $\underline{\gamma}^l/\bar{\beta}^l$ if and only if $p^{\bar{\beta}}y^{\underline{\gamma}} - w^{\bar{\beta}}x^{\underline{\gamma}} \leqslant p^{\bar{\beta}}y^{\bar{\beta}} - w^{\bar{\beta}}x^{\bar{\beta}}$ for at least one of the alternative (possibly multiple) solutions for $(y^{\underline{\gamma}}, -x^{\underline{\gamma}})$ and $(p^{\bar{\beta}}, w^{\bar{\beta}})$.*

If the necessary and sufficient condition for $\underline{\alpha}^l = \underline{\gamma}^l/\bar{\beta}^l$ specified above does not hold, then we need to resort to a computational solution of the nonlinear program in [P2] to obtain a numerical value for $\underline{\alpha}^l$. In the Appendix, we present some observations about the constraints in 2(a)–2(h) that become redundant for solving [P2] under these conditions. This reduction in the number of constraints is very important for applied work because it can significantly reduce the computational time to solve the nonlinear program.

Finally, we note that we have assumed observed profit to be positive in this section merely to facilitate analysis in terms of profit ratios. We conclude this section by alternatively defining efficiency measures in terms of profit variances (i.e., differences). An analogous set of definitions and propositions can be obtained thus to accommodate the more general case when observed profit may not be positive. That is, we define the aggregate and technical efficiency variances of observation l, relative to the set $T \in \mathscr{A}$ as:

$$\gamma_T^l \equiv \min\left\{\left(p^l y^l - w^l x^l\right) - \left(p^l y - w^l x\right) \middle| (y, -x) \in T\right\}, \quad \text{and}$$

$$\beta_T^l \equiv \min\left\{\left(p^l y^l - w^l x^l\right) - \left(p^l y - w^l x\right) \middle| (y, -x) \in T, \ y = y^l, \ x = hx^l\right\}.$$

Allocative efficiency variance is then defined as $\alpha_T^l \equiv \gamma_T^l - \beta_T^l$. It is evident that Proposition 3 continues to apply, and that the linear programs for determining $\underline{\gamma}^l$, $\bar{\gamma}^l$, $\underline{\beta}^l$, and $\bar{\beta}^l$ remain unchanged. In fact, $\underline{\gamma}^l = \Delta^l$. Similarly, Proposition 4 continues to hold after rewriting the objective functions of the nonlinear programs described in its statement, in terms of profit variances instead of ratios. The analog to Proposition 5 yields $\bar{\alpha}^l$ as equal to $\min\{0, \bar{\gamma}^l - \underline{\beta}^l\}$. Further, the condition identified in Proposition 6 determines whether $\underline{\alpha}^l = \underline{\gamma}^l - \bar{\beta}^l$.

4. AN EMPIRICAL ILLUSTRATION

In this section we illustrate the nonparametric analysis of technical, allocative, and aggregate efficiencies in production developed in this paper by employing a data set[14] obtained from a division of a large decentralized U.S. manufacturing firm. The data consists of twenty quarterly observations on prices and quantities of one output and three inputs (labor, materials, and capital) for the period from 1979 to 1983. Although a time series data set may not be the best context for analyzing efficiencies in production, it does permit an illustration of what can be accomplished by our method in terms of computing upper and lower bounds on technical, allocative, and aggregate efficiencies, when all production possibility sets satisfying the admissibility criteria are considered in the analysis.

The data set is reported in Tables I and II, and the bounds on the efficiency measures resulting from our analysis are reported in Table III. As noted in Section 3, we imposed additional restrictions on the admissible sets in order to bound the referent production possibility sets within the observed range of inputs and outputs. The input and output values for the referent points for evaluating the technical, allocative, and aggregate efficiencies were restricted to lie within the minimum and maximum observed values for each input and output. Using our criterion developed in Section 2, only two observations (#2 and #13) were found to belong to the efficient subset $\mathscr{E}$, and therefore, only these two have aggregate efficiency ratings of one.

It is interesting to note that the difference $(\bar{\gamma} - \underline{\gamma})$ between the upper and lower bounds for the aggregate efficiency measure has a small median value of 0.0026 (a

[14] See Banker and Datar (1987) for a detailed discussion. The data were collected by a direct analysis of original cost reports and documents to minimize measurement or recording errors. However, because of the aggregation into just three input categories, specification errors may arise.

TABLE I

OUTPUT-INPUT DATA

Obs. #	Output y	Input 1 x_1	Input 2 x_2	Input 3 x_3
1	94593	30722.0	38054.0	8184.00
2	95921	28365.0	35795.0	8119.00
3	76852	25445.0	31814.0	8079.00
4	94141	30648.9	41743.4	8667.62
5	102132	33279.0	41364.8	8881.57
6	100341	30828.1	40124.9	8744.03
7	81755	27360.1	32910.5	8109.84
8	95154	31544.9	39720.8	9659.96
9	91393	33485.8	40893.9	8889.40
10	90752	30725.8	39137.7	8808.96
11	75033	27881.6	32143.4	8442.41
12	85681	30042.7	28737.6	8113.79
13	87399	24799.7	32198.4	6962.07
14	80469	27676.7	38023.4	6887.93
15	65009	25173.9	30527.5	7051.72
16	86443	28634.7	43111.2	8520.29
17	94454	28289.2	46075.6	7384.74
18	84361	26157.7	39393.2	7344.97
19	76176	23490.0	33694.0	7351.46
20	75775	23078.5	31686.8	7311.08

TABLE II

SCALED OUTPUT-INPUT PRICE DATA[a]

Obs. #	Output Price p	Input 1 Price w_1	Input 2 Price w_2	Input 3 Price w_3
1	1.00000	1.00000	1.00000	1.00000
2	1.00000	1.00000	1.00000	1.00000
3	1.00000	1.00000	1.00000	1.00000
4	1.03000	1.04686	1.01700	1.04700
5	1.03000	1.04688	1.01700	1.04700
6	1.03000	1.04684	1.01700	1.04700
7	1.03000	1.04689	1.01700	1.04700
8	1.06200	1.09571	1.04600	1.09400
9	1.06200	1.09593	1.04600	1.09400
10	1.06200	1.09553	1.04600	1.09400
11	1.06200	1.09628	1.04600	1.09400
12	1.10000	1.15742	1.10900	1.16000
13	1.10000	1.15682	1.10900	1.16000
14	1.10000	1.15556	1.10900	1.16000
15	1.10000	1.15397	1.10900	1.16000
16	1.12500	1.22135	1.12400	1.23200
17	1.12500	1.22015	1.12400	1.23200
18	1.12500	1.21861	1.12400	1.23200
19	1.12500	1.21992	1.12400	1.23200
20	1.19800	1.27937	1.19400	1.31800

[a] The actual prices are scaled in terms of period 1 prices such that the output and input quantities for period 1 in Table I correspond to the actual monetary values.

TABLE III

COMPUTED EFFICIENCIES (UPPER & LOWER BOUNDS)

Obs. #	Aggregate Efficiency Upper Bound $\bar{\gamma}$	Aggregate Efficiency Lower Bound $\underline{\gamma}$	Technical (Farrell) Upper Bound $h^{\bar{\beta}}$	Technical (Farrell) Lower Bound $h^{\underline{\beta}}$	Technical Efficiency Upper Bound $\bar{\beta}$	Technical Efficiency Lower Bound $\underline{\beta}$	Allocative Efficiency Upper Bound $\bar{\alpha}$	Allocative Efficiency Lower Bound $\underline{\alpha}$
1	0.7458	0.7458	0.9618	0.9219	0.8571	0.7458	1.0000	0.8700 *
2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000 *
3	0.4870	0.4870	0.9728	0.9070	0.8666	0.6545	0.7440	0.5619 *
4	0.5517	0.5492	0.9028	0.8702	0.6218	0.5517	1.0000	0.9251
5	0.7847	0.7811	1.0000	0.9397	1.0000	0.7851	0.9995	0.7811 *
6	0.8726	0.8686	1.0000	0.9624	1.0000	0.8722	1.0000	0.8685 *
7	0.5621	0.5596	0.9386	0.8732	0.7585	0.6031	0.9321	0.7376 *
8	0.5875	0.5815	0.8919	0.8837	0.6054	0.5877	0.9997	0.9681
9	0.3214	0.3181	0.8413	0.8143	0.3570	0.3218	0.9988	0.9390
10	0.4962	0.4912	0.8572	0.8541	0.5024	0.4971	0.9983	0.9853
11	0.2561	0.2534	0.9409	0.8940	0.5909	0.4459	0.5743	0.4288 *
12	0.7693	0.7693	1.0000	1.0000	1.0000	1.0000	0.7693	0.7692 *
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000 *
14	0.2691	0.2690	1.0000	1.0000	1.0000	1.0000	0.2691	0.2689 *
15	0.0179	0.0179	1.0000	0.9768	1.0000	0.2047	0.0875	0.0179 *
16	0.1428	0.1413	0.8611	0.8084	0.2029	0.1558	0.9162	0.6962 *
17	0.4660	0.4614	1.0000	0.9327	1.0000	0.6285	0.7415	0.4698
18	0.4158	0.4117	0.9479	0.9378	0.6863	0.6467	0.6430	0.5998 *
19	0.4340	0.4297	0.9897	0.9825	0.9290	0.8842	0.4908	0.4625 *
20	0.5436	0.5380	1.0000	1.0000	1.0000	1.0000	0.5436	0.5379 *

Note: The lower bound allocative efficiencies labelled with an * could be computed simply as $\underline{\gamma}/\bar{\beta}$ since the condition in Proposition 6 was satisfied for these points. The remaining five lower bounds required a computational solution of the nonlinear program.

mean value of 0.0025), and the largest difference is only 0.0060. These differences are small because the relative price vectors of the two observations (#2 and #13) are not considerably different. Therefore, for any $l \in \mathscr{I}, (p^l y - w^l x)$ is not much greater when the referent point for aggregate efficiency evaluation is chosen from the largest admissible set L than when it is chosen from the smallest admissible set S. The expansion of the space from which the respective referent points may be chosen when one moves from the smallest set S to the largest set L has, however, a greater impact on our technical and allocative efficiency measures. This is witnessed by the fact that $(\bar{\beta} - \underline{\beta})$ and $(\bar{\alpha} - \underline{\alpha})$ have median values of 0.046 and 0.0697, mean values of 0.1197 and 0.0910, and maximum values of 0.7953 and 0.2717, respectively.

In interpreting the results it must be remembered that our efficiency measures are based on profit ratios. One consequence of this is that, with respect to any admissible set, our technical efficiency measure β will always be less than or equal to the Farrell-type radial measure h^{β}. Further, any reduction in h^{β} is magnified when translated into profit terms, that is, $(h^{\bar{\beta}} - h^{\underline{\beta}})$ is always less than or equal to $(\bar{\beta} - \underline{\beta})$. We focus on observation #15 as it illustrates the above remark. At first glance, it may appear surprising that the aggregate efficiency drops from 1.0 to 0.0179 over a span of three quarters (from the 13th to the 15th). However, note that the inputs and prices of observation #15 are not very different from those of observation #13 (which is the aggregate efficiency referent point for observation #15), but the output for observation #15 is much lower than that for observation #13. Therefore, measuring in terms of quarter 15 prices, the actual profits in quarter 15 are only 429.0 (in scaled units) compared to 23736.8 (in scaled units) for quarter 13, and its aggregate[15] efficiency rating is 0.0179 (= 429.0/23736.8). Since our technical efficiency measure is input based, it is not immediately clear, however, that observation #15 is technically inefficient. It is possible that an increase of approximately 1650 units of the second input (i.e., $x_2^{13} - x_2^{15}$) yields a sizable increase in output, notwithstanding modest decreases in other inputs. Indeed our analysis indicates that observation #15 could be technically efficient or inefficient, but it is unambiguously allocatively inefficient. This emphasizes the contribution of our nonparametric analysis; it reveals when, without assuming a restrictive and untestable parametric functional form, one can identify inefficiency within tight bounds, and also when one cannot do so.

It is also of interest to note that in all but five of the twenty cases the condition of Proposition 6 was satisfied by the solutions obtained for γ and $\bar{\beta}$. Therefore, for these fifteen cases $\underline{\alpha}$ could be computed simply as $\gamma/\bar{\beta}$. For the remaining five cases $\underline{\alpha}$ was determined by a computational solution to the nonlinear programming problem [P2C],[16] using a VAX/VMS implementation of the code NPSOL distributed by the Stanford Optimization Laboratory. With an objective function

[15] Note that the Farrell-type radial technical efficiency measure for observation #15 is bounded between 0.9768 and 1.0.

[16] Problem [P2C] is a relaxed version of problem [P2], and is described in the Appendix.

accuracy of 10^{-6} and constraint tolerances of the order of the data accuracy (10^{-5}), CPU time was in the range of 2 to 4 seconds for each problem.

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Manuscript received February, 1986; final revision received October, 1987.

APPENDIX

PROOF OF PROPOSITION 1: First we show that $\mathscr{E}$ is subset rationalizable. Since $\Delta_i = 0$ for $i \in \mathscr{E}$, we have $p^i y^i - w^i x^i \geqslant p^i y^j - w^i x^j$ for all $j \in \mathscr{I}$. It is apparent then that the production possibility set $T = \{(y^i, -x^i) \mid i \in \mathscr{I}\}$ subset rationalizes $\mathscr{E}$.

Next we consider a subset $\mathscr{H} \subseteq \mathscr{I}$ that is subset rationalized by some production set T'. Then, by the definition of subset rationalization, for every $l \in \mathscr{H}$, $p^l y^l - w^l x^l \geqslant p^l y - w^l x$ for all $(y, -x) \in T'$, and further $(y^j, -x^j) \in T'$ for all $j \in \mathscr{I}$. Hence, we have for every $l \in \mathscr{H}$, $p^l y^l - w^l x^l \geqslant p^l y^j - w^l x^j$ for all $j \in \mathscr{I}$. This implies that $\Delta_l = 0$ for all $l \in \mathscr{H}$, and therefore $\mathscr{H} \subseteq \mathscr{E}$. *Q.E.D.*

PROOF OF PROPOSITION 2: Evidently, S is a closed and convex set, and $(y^i, -x^i) \in S$ for all $i \in \mathscr{I}$. Consider next an observation $l \in \mathscr{E} \equiv \{i \mid \Delta_i = 0\}$ so that $\Delta_l = 0$. Therefore, $p^l y^l - w^l x^l \geqslant p^l y^i - w^l x^i$ for all $i \in \mathscr{I}$.

If $(y, -x) \in S$, then there exist some $\lambda_i \geqslant 0$ with $\Sigma_{i \in \mathscr{I}} \lambda_i = 1$, such that $y \leqslant \Sigma_{i \in \mathscr{I}} \lambda_i y^i$ and $x \geqslant \Sigma_{i \in \mathscr{I}} \lambda_i x^i$. Therefore,

$$p^l y - w^l x \leqslant p^l \sum_{i \in \mathscr{I}} \lambda_i y^i - w^l \sum_{i \in \mathscr{I}} \lambda_i x^i = \sum_{i \in \mathscr{I}} \lambda_i (p^l y^i - w^l x^i) \leqslant p^l y^l - w^l x^l.$$

Thus, the closed convex set S weakly rationalizes the observed data. *Q.E.D.*

The following two corollaries to the definitions of aggregate, technical, and allocative efficiencies are required for proofs of subsequent propositions.

COROLLARY 1: *The point* $(y^\gamma, -x^\gamma)$ *solves the problem in Definition* 5, *and is hence the technically and allocatively efficient referent point for observation* l *relative to the set* $T \in \mathscr{A}$, *if and only if the hyperplane* $p^l y - w^l x = p^l y^\gamma - w^l x^\gamma$ *supports the convex set* T *at* $(y^\gamma, -x^\gamma)$.

COROLLARY 2: *The point* $(y^\beta, -x^\beta)$ *solves the problem in Definition* 6, *and is hence the technically efficient referent point for observation* l *relative to the set* $T \in \mathscr{A}$, *if and only if there exists some nonnegative price vector* $(p^\beta, w^\beta) \neq 0$, *such that the hyperplane* $p^\beta y - w^\beta x = p^\beta y^\beta - w^\beta x^\beta$ *supports the convex set* T *at* $(y^\beta, -x^\beta)$.

The proofs of the above corollaries follow simply by implication of the admissibility criteria for T, and the fact that in order to solve the respective problems, the points $(y^\gamma, -x^\gamma)$ and $(y^\beta, -x^\beta)$ must lie on the boundary of T.

PROOF OF LEMMA: It is evident from Proposition 2 that $S \in \mathscr{A}$. It is also straightforward to verify that L, determined as the intersection of half-spaces, is also admissible.

If $T \in \mathscr{A}$, then conditions 1(b), (c), and (d) imply that T satisfies Postulates 1, 2, and 3 of Banker, Charnes, and Cooper (1984) described in Section 2. Since S is the set determined by their four postulates, it follows from Postulate 4 that $S \subseteq T$.

Further, since T (and L) $\in \mathscr{A}$, it follows from condition 1(a) that $T \subseteq L$. *Q.E.D.*

PROOF OF PROPOSITION 3: Note that in each case we have a bi-extremal optimization problem with an embedded minimization operation specified in Definitions 5 or 6. The proof is evident since from the preceding lemma, we have $S \subseteq T \subseteq L$ for all $T \in \mathscr{A}$. *Q.E.D.*

PROOF OF PROPOSITION 4: We first prove the proposition about the upper bound $\bar{\alpha}^l$. From Definition 7, $\bar{\alpha}^l = \max_{T \in \mathscr{A}} \{\gamma_T^l/\beta_T^l\}$, and from the definitions of γ_T^l and β_T^l it follows that $\bar{\alpha}^l$ is given by:

$$[\text{P3}]: \qquad \bar{\alpha}^l = \max_{T \in \mathscr{A}} \left(p^l y^\beta - w^l x^\beta \right) \Big/ \left(p^l y^\gamma - w^l x^\gamma \right)$$

subject to

$$\left(y^\beta, -x^\beta \right) \in \operatorname{argmax} \left\{ \left(p^l y^\beta - w^l x^\beta \right) \middle| \left(y^\beta, -x^\beta \right) \in T, y^\beta = y^l, x^\beta = hx^l, h \geqslant 0 \right\},$$

$$\left(y^\gamma, -x^\gamma \right) \in \operatorname{argmax} \left\{ \left(p^l y^\gamma - w^l x^\gamma \right) \middle| \left(y^\gamma, -x^\gamma \right) \in T \right\}.$$

Note that problems [P1] and [P3] have the same objective function.

Now consider an optimal solution $T^*, y^{\beta^*}, x^{\beta^*}, y^{\gamma^*}, x^{\gamma^*}, p^{\beta^*}, w^{\beta^*}, h^*$ to the problem [P3]. By the definition of $\mathscr{A}$, $T^* \in \mathscr{A}$ weakly rationalizes the observed data $\mathscr{J}$, and hence the above solution satisfies constraints 2(b) and 2(d). Also by the definition of $\mathscr{A}$, $(y^j, -x^j) \in T^*$ for all $j \in \mathscr{J}$. Hence the solution satisfies constraints 2(a) and 2(g). By the definition of the technical efficiency measure, constraints 2(e) and 2(f) are satisfied. Further, as noted earlier, if $(y^{\beta^*}, -x^{\beta^*})$ is a technically efficient referent point, there must exist some nonnegative price vector $(p^{\beta^*}, w^{\beta^*})$, $w^{\beta^*} \neq 0$, such that the hyperplane $p^{\beta^*}y - w^{\beta^*}x = p^{\beta^*}y^{\beta^*} - w^{\beta^*}x^{\beta^*}$ supports the convex set T at the point $(y^{\beta^*}, -x^{\beta^*})$. Thus the optimal solution to [P3] also satisfies constraints 2(c), 2(h), and 2(i).

Since the solution to problem [P3] satisfies constraints set 2(a)–2(i), it follows that $\bar{\alpha}^l \leqslant \bar{\psi}^l$. In order to complete the proof that $\bar{\alpha}^l = \bar{\psi}^l$, we next establish that $\bar{\alpha}^l \geqslant \bar{\psi}^l$. To do this, consider an optimal solution $y^{\beta^0}, x^{\beta^0}, y^{\gamma^0}, x^{\gamma^0}, p^{\beta^0}, w^{\beta^0}, h^0$ to the problem [P1]. We shall establish that this is a feasible solution to the problem [P3]. This is done by constructing an admissible production possibility set $T \in \mathscr{A}$, with respect to which $(y^{\beta^0}, -x^{\beta^0})$ and $(y^{\gamma^0}, -x^{\gamma^0})$ are the referent points for observation l for evaluating its technical and aggregate efficiency respectively.

Consider the production possibility set $T = \{(y, -x) | (y, -x) \text{ satisfy conditions 3(a)–3(c)}\}$ where conditions 3(a)–3(c) are specified as follows:

$$3(\text{a}) \qquad p^l y - w^l x \leqslant p^l y^{\gamma^0} - w^l x^{\gamma^0},$$

$$3(\text{b}) \qquad p^{\beta^0} y - w^{\beta^0} x \leqslant p^{\beta^0} y^{\beta^0} - w^{\beta^0} x^{\beta^0},$$

$$3(\text{c}) \qquad p^i y - w^i x \leqslant p^i y^i - w^i x^i \quad \forall i \in \mathscr{E}.$$

Obviously by virtue of 3(c), T satisfies property 1(a) of admissible production possibility sets. By virtue of 2(a), 2(c), 3(a), and 3(b) it is clear that $(y^j, -x^j) \in T$ for all $j \in \mathscr{J}$. Thus T also satisfies the admissibility condition 1(b). Since T is the intersection of half spaces, it is closed and convex, and thus satisfies admissibility condition 1(c). Further since T obviously satisfies admissibility condition 1(d), it follows that $T \in \mathscr{A}$.

Next note that by virtue of 2(e) and 2(f), and the fact that by virtue of 3(b) $p^{\beta^0}y - w^{\beta^0}x = p^{\beta^0}y^{\beta^0} - w^{\beta^0}x^{\beta^0}$ is a supporting hyperplane to T at the point $(y^{\beta^0}, -x^{\beta^0})$, it follows from Corollary 2 that $(y^{\beta^0}, -x^{\beta^0})$ is the technically efficient referent point for observation l with respect to the production possibility set T. Similarly from 3(a) and Corollary 1 it is obvious that $(y^{\gamma^0}, -x^{\gamma^0})$ is the technically and allocatively efficient referent point for observation l with respect to the production possibility set T. Therefore, an optimal solution to the problem [P1] is feasible for the problem [P3], and hence it follows that $\bar{\alpha}^l \geqslant \bar{\psi}^l$. Since $\bar{\alpha}^l \leqslant \bar{\psi}^l$ and $\bar{\alpha}^l \geqslant \bar{\psi}^l$, it follows that $\bar{\alpha}^l = \bar{\psi}^l$.

A similar proof establishes that $\underline{\alpha}^l = \underline{\psi}^l$. *Q.E.D.*

PROOF OF PROPOSITION 5: We shall first prove that if $p^l y^{\underline{\beta}} - w^l x^{\underline{\beta}} < p^l y^j - w^l x^j$ for some $j \in \mathscr{J}$, then $\bar{\alpha}^l = \bar{\gamma}^l/\underline{\beta}^l$. Obviously since $\bar{\alpha}^l \leqslant \bar{\gamma}^l/\underline{\beta}^l$ this can be done by showing that if $p^l y^{\underline{\beta}} - w^l x^{\underline{\beta}} < p^l y^j - w^l x^j$ for some $j \in \mathscr{J}$, then $y^{\underline{\beta}}, x^{\underline{\beta}}, y^{\bar{\gamma}}, x^{\bar{\gamma}}, p^{\underline{\beta}}, w^{\underline{\beta}}$, is a feasible solution[17] to problem [P1]. Clearly, the

[17] Here $(y^\gamma, -x^\gamma)$ is the referent output-input vector for evaluating the aggregate efficiency of observation l with respect to the smallest admissible set S.

definitions of admissible production possibility sets, and aggregate and technical efficiencies imply that constraints 2(a) to 2(f) are satisfied. Since $\bar{\gamma}^l$ is attained for the smallest admissible production possibility set S, which is the monotone convex hull of the observed output-input vectors, $(y^{\bar{\gamma}}, -x^{\bar{\gamma}}) = (y^j, -x^j)$ for some $j \in \mathscr{J}$. It follows from constraints 2(c) that constraint 2(h) is satisfied. Constraint 2(g) is seen to hold by virtue of the condition $p^l y^{\underline{\beta}} - w^l x^{\underline{\beta}} < p^l y^j - w^l x^j$ for some $j \in \mathscr{J}$, in conjunction with constraint 2(a). Thus if $p^l y^{\underline{\beta}} - w^l x^{\underline{\beta}} < p^l y^j - w^l x^j$ for some $j \in \mathscr{J}$, $y^{\underline{\beta}}, x^{\underline{\beta}}, p^{\underline{\beta}}, w^{\underline{\beta}}, y^{\bar{\gamma}}, x^{\bar{\gamma}}$, is a feasible solution to the problem [P1] and consequently $\bar{\alpha}^l = \bar{\gamma}^l / \underline{\beta}^l$.

Finally we note that if $p^l y^{\underline{\beta}} - w^l x^{\underline{\beta}} \geqslant p^l y^j - w^l x^j$ for all $j \in \mathscr{J}$ we can set $y^{\gamma} = y^{\beta} = y^{\underline{\beta}}$ and $x^{\gamma} = x^{\beta} = x^{\underline{\beta}}$ in the constraints 2(a) to 2(h) and get $\bar{\alpha}^l = 1$. *Q.E.D.*

Proof of Proposition 6: If $\underline{\alpha}^l = \underline{\gamma}^l / \bar{\beta}^l$, then $y^{\underline{\gamma}}, x^{\underline{\gamma}}, y^{\bar{\beta}}, x^{\bar{\beta}}, p^{\bar{\beta}}, w^{\bar{\beta}}$ is an optimal solution to the nonlinear programming problem in [P2]. Therefore, constraint 2(h) holds for these values of γ and β, and hence $p^{\bar{\beta}} y^{\underline{\gamma}} - w^{\bar{\beta}} x^{\underline{\gamma}} \leqslant p^{\bar{\beta}} y^{\bar{\beta}} - w^{\bar{\beta}} x^{\bar{\beta}}$.

To prove the converse we first note that $\underline{\alpha}^l \geqslant \underline{\gamma}^l / \bar{\beta}^l$. Next we proceed to show that under the theorem condition $y^{\underline{\gamma}}, x^{\underline{\gamma}}, y^{\bar{\beta}}, x^{\bar{\beta}}, p^{\bar{\beta}}, w^{\bar{\beta}}$ satisfy the constraints in 2(a)–2(i) and therefore are feasible for the minimization problem. This will establish $\underline{\alpha}^l \leqslant \underline{\gamma}^l / \bar{\beta}^l$ and hence $\underline{\alpha}^l = \underline{\gamma}^l / \bar{\beta}^l$.

To show that $y^{\underline{\gamma}}, x^{\underline{\gamma}}, y^{\bar{\beta}}, x^{\bar{\beta}}, p^{\bar{\beta}}, w^{\bar{\beta}}$ is a feasible solution we first note the fact that constraints 2(a) to 2(f) are satisfied, which is evident from the definitions of $(y^{\underline{\gamma}}, -x^{\underline{\gamma}})$ as the solution for aggregate efficiency evaluation relative to the set L, and of $(y^{\bar{\beta}}, -x^{\bar{\beta}})$ as the solution for technical efficiency evaluation relative to the set S. The satisfaction of constraint 2(h) is evident from the theorem condition. Finally, we note that $(y^{\bar{\beta}}, -x^{\bar{\beta}}) \in S$, and therefore there exist some $\lambda_j \geqslant 0$, with $\sum_{j \in \mathscr{I}} \lambda_j = 1$, $\mathscr{I} \subseteq \mathscr{J}$, such that $y^{\beta} \leqslant \sum_{j \in \mathscr{I}} \lambda_j y^j$ and $x^{\beta} \geqslant \sum_{j \in \mathscr{I}} \lambda_j x^j$. Therefore, from the constraint 2(a) we have

$$\begin{aligned} p^l y^{\underline{\gamma}} - w^l x^{\underline{\gamma}} &\geqslant \sum_{j \in \mathscr{I}} \lambda_j \left(p^l y^j - w^l x^j \right) \\ &\geqslant p^l y^{\bar{\beta}} - w^l x^{\bar{\beta}}. \end{aligned}$$

Thus, constraint 2(g) is also satisfied and hence $y^{\underline{\gamma}}, x^{\underline{\gamma}}, y^{\bar{\beta}}, x^{\bar{\beta}}, p^{\bar{\beta}}, w^{\bar{\beta}}$ is a feasible solution to the minimization problem in [P2]. Therefore, $\underline{\alpha}^l = \underline{\gamma}^l / \bar{\beta}^l$. *Q.E.D.*

Reduction of constraints 2(a)–2(h) for computing $\underline{\alpha}^l$: First we note that the constraints in 2(a) are redundant. This is because any $(y^j, -x^j)$, $j \in \mathscr{J}$, is feasible as a candidate $(y^{\gamma}, -x^{\gamma})$ with respect to the remaining constraints. Therefore an optimal solution to the relaxed problem [P2A], obtained by deleting the constraints in 2(a) from the problem [P2], will necessarily satisfy the deleted constraints, or else it can be improved upon for solving [P2A] by substituting the specific $(y^j, -x^j)$ in the violated constraint for $(y^{\gamma^*}, -x^{\gamma^*})$. Thus the optimal solution to the relaxed problem [P2A] will also be optimal for the original problem [P2].

Next we note that the constraint 2(g) is also redundant. This is because any $(y^{\beta}, -x^{\beta})$ with $y^{\beta}, x^{\beta} \geqslant 0$ is feasible as a candidate $(y^{\gamma}, -x^{\gamma})$ for the further relaxed problem [P2B] obtained by deleting the constraint 2(g) from the problem [P2A]. That is, any solution to the problem [P2B] that violates constraint 2(g) can be improved upon for solving [P2B] by substituting $(y^{\beta^*}, -x^{\beta^*})$ for $(y^{\gamma^*}, -x^{\gamma^*})$.

Now consider the problem [P2C] obtained by deleting the linear constraints 2(d) from the problem [P2B]. If a solution $(p^{\beta^*}, w^{\beta^*}, h^*)$ to the problem [P2C] is such that it violates constraints 2(d) for some $i \in \mathscr{E}$, then it can be improved upon by substituting (p^i, w^i, h_i) where $h_i = (p^i y^l - p^i y^i + w^i x^i)/(w^i x^l)$, noting that the latter solution is indeed feasible for the problem [P2C]. All of the above imply that we need to obtain a solution to only the relaxed problem [P2C] to compute $\underline{\alpha}^l$. Nevertheless, problem [P2C] remains a nonlinear program.

REFERENCES

Afriat, S. (1972): "Efficiency Estimates of Production Functions," *International Economic Review*, 13, 568–598.

Banker, R. D. (1980): "Studies in Cost Allocation and Efficiency Evaluation," unpublished doctoral thesis, Harvard University, Graduate School of Business Administration.

——— (1984): "Estimating Most Productive Scale Size Using Data Envelopment Analysis," *European Journal of Operational Research*, 17, 35–44.

——— (1987): "Maximum Likelihood, Consistency and Data Envelopment Analysis," Carnegie Mellon University, Working Paper.

BANKER, R. D., A. CHARNES, AND W. W. COOPER (1984): "Models for the Estimation of Technical and Scale Inefficiencies in Data Envelopment Analysis," *Management Science*, 30, 1078–1092.

BANKER, R. D., A. CHARNES, W. W. COOPER, AND A. SCHINNAR (1981): "A Bi-Extremal Principle for Frontier Estimation and Efficiency Evaluation," *Management Science*, 27, 1370–1382.

BANKER, R. D., R. F. CONRAD, AND R. P. STRAUSS (1986): "A Comparative Application of DEA and Translog Methods: An Illustrative Study of Hospital Production," *Management Science*, 32, 30–44.

BANKER, R. D., AND S. DATAR (1987): *Productivity Accounting for Manufacturing Operations*, research monograph commissioned by the American Accounting Association, in process.

BANKER, R. D., AND A. MAINDIRATTA (1986): "Piecewise Loglinear Estimation of Efficient Production Surfaces," *Management Science*, 32, 126–135.

BANKER, R. D., AND R. MOREY (1986a): "Efficiency Analysis for Exogeneously Fixed Inputs and Outputs," *Operations Research*, 34, 513–521.

——— (1986b): "Data Envelopment Analysis with Categorical Inputs and Outputs," *Management Science*, 32, 1613–1627.

CAVES, D. W., AND L. R. CHRISTENSEN (1980): "Global Properties of Flexible Functional Forms," *American Economic Review*, 70, 422–432.

CHARNES, A., W. W. COOPER, AND E. RHODES (1978): "Measuring the Efficiency of Decision Making Units," *European Journal of Operational Research*, 11, 429–444.

——— (1981): "Data Envelopment Analysis: Approach for Evaluating Program and Managerial Efficiency," *Management Science*, 27, 668–697.

DIEWERT, W. E., AND C. PARKAN (1984): "Linear Programming Tests of Regularity Conditions for Production Functions," in *Quantitative Studies on Production and Prices*, ed. by W. Eichhorn, K. Henn, K. Neumann, and R. Shephard. Physica-Verlag.

FARRELL, M. J. (1957): "The Measurement of Productive Efficiency," *Journal of the Royal Statistical Society*, Series A, 120, 253–290.

HANOCH, G., AND M. ROTHSCHILD (1972): "Testing the Assumptions of Production Theory: A Nonparametric Approach," *Journal of Political Economy*, 79, 256–275.

LOVELL, C. A. K., AND P. SCHMIDT (1982): "A Comparison of Alternative Approaches to the Measurement of Productive Efficiency," presented at the Fourth Annual Conference on Current Issues in Productivity, Ithaca, N.Y.

SAMUELSON, P. (1947): *Foundations of Economic Analysis*. Cambridge: Harvard University Press.

VARIAN, H. (1984): "The Nonparametric Approach to Production Analysis," *Econometrica*, 54, 579–597.