ELSEVIER

European Journal of Operational Research 88 (1996) 583–585

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Theory and Methodology

A note on returns to scale in DEA

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Received March 1994; revised September 1994

Abstract

This brief note adds computational convenience and efficiency to the article by Banker and Thrall on returns to scale in DEA by modifying one of their suggestions to avoid the need for examining *all* alternate optima in order to reach a decision.

Keywords: Data Envelopment Analysis; Returns to scale; Frontier projection operator

1. Background

The standard model for analyzing returns-to-scale in DEA was first addressed explicitly in the Harvard Business School thesis of Banker (1980) where it was formulated in terms of the following dual pair of linear programming problems:

$$\text{Min}\quad \theta_o - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_i^+\right) \tag{1}$$

subject to

$$0 = \theta_o - \sum_{i=1}^{n} x_{ij}\lambda_j - s_i^-,$$

$$y_{ro} = \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+,$$

$$1 = \sum_{j=1}^{n} \lambda_j, \quad 0 \le \lambda_j,\ s_i^-,\ s_r^+,$$

$$\text{Max}\quad \sum_{r=1}^{s} \mu_r y_{r0} + u_0 \tag{2}$$

subject to

$$-\sum_{i=1}^{m} \nu_i x_{ij} + \sum_{r=1}^{s} \mu_r y_{rj} + u_o \le 0,$$

$$\sum_{i=1}^{m} \nu_i x_{io} = 1,$$

$$-\nu_i \le -\varepsilon,$$

$$-\mu_r \le -\varepsilon,$$

where:
x_{ij} = Observed amount of input i for DMU_j,
y_{rj} = Observed amount of output r for DMU_j,
with $i = 1,\dots, m$; $r = 1,\dots, s$; $j = 1,\dots, n$; and DMU_o represents one of the DMU_j (= Decision Making Units) whose input–output record is to be evaluated for its efficiency relative to the performance of all DMU_j (including itself).

As described by Banker (1980), the focus of the returns-to-scale analysis was on the sign of

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SSDI 0377-2217(94)00281-9

u_o. This was extended in Banker, Charnes and Cooper (1984) where unique solutions were explicitly assumed. Continuing in this tradition, Banker (1984) introduced the important new concept of mpss (most productive scale size). Formally, mpss occurs when (xX_o, yY_o) with $0 \le x \le y$ for augmentations of output in association with input augmentation in the neighborhood of (X_o, Y_o). Here X_o and Y_o are the input and output vectors for DMU_o while x and y are scalars. Informally, expansions beyond mpss will have at least one output increasing at a rate that is less than proportionate to the rate at which all inputs are increased.

The above developments with respect to returns to scale in DEA are all made precise in Banker and Thrall (1992), who also extend the analysis to allow for alternate optima, for which they give the following characterizations:

I. Increasing returns-to-scale prevail at (X_o, Y_o) if and only if $u_o^* > 0$ for *all* optimal solutions.
II. Decreasing returns-to-scale prevail at (X_o, Y_o) if and only if $u_o^* < 0$ for *all* optimal solutions.
III. Constant returns-to-scale prevail at (X_o, Y_o) if $u_o^* = 0$ in *any* optimal solution.

Banker and Thrall also provide a measure of scale elasticity and a model for determining its bounds which we can modify, as is done in the following section, when interest centers only on the returns to scale depicted above. See Cooper, Thompson and Thrall (1996) for detailed discussions and interpretations.

2. Projections

Returns-to-scale has an unambiguous meaning only if the point (X_o, Y_o) is on the efficiency frontier. In an approach for bounding the scale elasticities suggested by Banker and Thrall (1992), it is assumed that (X_o, Y_o) is on the efficiency frontier but we can eliminate the need for this assumption by using the following projections, as given in Charnes, Cooper and Rhodes (1978): [1]

$$\hat{x}_{io} = \theta_o^* x_{io} - s_i^{-*}, \quad i = 1, \dots, m,$$
$$\hat{y}_{ro} = y_{ro} + s_r^{+*}, \quad r = 1, \dots, s, \tag{3}$$

where $\hat{x}_{io}$, $\hat{y}_{ro}$ are new values for the corresponding inputs and outputs with $\hat{x}_{io} \le x_{io}$ and $\hat{y}_{ro} \ge y_{ro}$ as obtained from optimal values for the variables θ_o^*, s_i^{-*}, s_r^{+*}.

To avoid the need for exploring *all* alternate optima, we can proceed as follows. Suppose we have an optimum for (1)–(2) with $u_o^* < 0$. We then solve

$$\text{Max} \quad u_o \tag{4}$$

subject to

$$-\sum_{i=1}^{m} \nu_i x_{ij} + \sum_{r=1}^{s} \mu_r y_{rj} + u_o \le 0,$$
$$j = 1, \dots, n, \ j \ne o,$$
$$-\sum_{i=1}^{m} \nu_i \hat{x}_{io} + \sum_{r=1}^{s} \mu_r \hat{y}_{ro} + u_o \le 0, \quad j = o,$$
$$\sum_{i=1}^{m} \nu_i \hat{x}_{io} = 1,$$
$$\sum_{r=1}^{s} \mu_r \hat{y}_{ro} + u_o = 1,$$
$$u_o \le 0,$$
$$\nu_i \ge \varepsilon,$$
$$\mu_r \ge \varepsilon,$$

where the values $\hat{x}_{io}$, $i = 1, \dots, m$, and $\hat{y}_{ro}$, $r = 1, \dots, s$, are obtained from (3). If, alternatively, we have a solution of (1)–(2) with $u_o^* > 0$ we replace $u_o \le 0$ in (4) with $u_o \ge 0$ and reorient the objective to Min u_o. In either case, a solution with a new $u_o^* = 0$ means that returns to scale are locally constant. Otherwise all alternate optima have the sign originally obtained from (1)–(2), and all possibilities are thus covered as required in characterizations I–III.

[1] For an up-to-date treatment of these and other projection operators in DEA, see Banker et al. (1994).

Use of the approach we have suggested can alleviate difficulties like those reported in Ganley and Cubbin (1992, p.111): "... the computational intractability of the BT procedure is both frustrating and disappointing... and have prohibited complete confirmation of [our] RTS results". Thus in their study of British prisons, GC use an 'intuitively appealing' procedure which can be more simply implemented with the rigorous one we have now supplied.

3. Concluding comments

Banker and Thrall (1992) omitted the non-Archimedean element $\varepsilon > 0$ in their formulation of (1)–(2) and hence did not allow for the possible presence of non-zero slack. Unless this is done [2] some care is needed because, with no provision for possible alternate optima, an optimum with

$$\theta_o^* = \sum_{r=1}^{s} \mu_r^* y_{ro} + u_o^* = 1$$

does not suffice to guarantee efficiency.

Our use of the projections given in (3) removes the need for assuming that efficiency was attained by the DMU being analyzed and it also eliminates the infinite solutions which Banker and Thrall note are possible with their formulations. There are, however, other problems. For instance, different results may be secured from output-oriented and input-oriented models when the corresponding projections are made for DMUs which did not perform efficiently [3] – although, of course, this problem does not arise if the DMU under analysis had performed efficiently, in accordance with the Banker and Thrall (1992) assumptions. Finally, we also need to note that different DEA models may also yield different results. See the comparison of 'additive', 'multiplicative' and other DEA models in Ahn, Charnes and Cooper (1988).

[2] Or unless some other approach is provided. Cf., e.g., Banker, Chang and Cooper (1996) for a use of CCR models to determine returns to scale.

[3] This suggests the need for new types of projection operators as outlined in Thrall (1996).

Acknowledgments

Support for the research reported in this paper has been provided by the IC2 Institute of the University of Texas at Austin. The authors are also grateful to Anand Desai and Lawrence Walters for correspondence which helped to stimulate the developments reported in this paper.

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